

# Classification of Möbius Homogeneous Wintgen Ideal Submanifolds

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## Abstract

A submanifold  $f : M^m \rightarrow \mathbb{Q}^{m+p}(c)$  in a real space form attaining equality in the DDVV inequality at every point is called a Wintgen ideal submanifold. They are invariant objects under the Möbius transformations. In this paper, we classify those Wintgen ideal submanifolds of dimension  $m \geq 3$  which are Möbius homogeneous. There are three classes of non-trivial examples, each related with a famous class of homogeneous minimal surfaces in  $S^n$  or  $\mathbb{C}P^n$ : the cones over Veronese surfaces  $S^2 \rightarrow S^n$ , the cones over homogeneous flat minimal surfaces  $\mathbb{R}^2 \rightarrow S^n$ , and the Hopf bundle over the Veronese embeddings  $\mathbb{C}P^1 \rightarrow \mathbb{C}P^n$ .

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## 1 Introduction

A central theme in geometry is to find and characterize those best shapes. It often means to find the optimally immersed submanifolds in a fixed ambient space. Two widely used optimality criteria are the minimization of certain functional(s), and the existence of many symmetries.

From this viewpoint, homogeneous minimal surfaces in real space forms are best submanifolds, which include the Veronese surfaces  $S^2 \rightarrow S^{2k}$  and Clifford type surfaces  $\mathbb{R}^2 \rightarrow S^{2k+1}$  [3, 15]. In complex space forms there are similar examples [1, 7].

In this paper we will consider Möbius homogeneous, Wintgen ideal submanifolds in Möbius geometry, which might be regarded also as best submanifolds according to both criteria. To our happy surprise, the classification shows that they are closely related with those homogeneous minimal surfaces mentioned above.

To explain what is a *Wintgen ideal submanifold*, note that an optimality criterion is to consider a universal inequality and find out all the cases when the equality is achieved, which is somewhat similar to the minimization criterion. In this spirit, we are interested in the equality case in the so-called *DDVV inequality* [10] for a generic submanifold  $f : M^m \rightarrow \mathbb{Q}^{m+p}(c)$  in a real space form. This inequality is remarkable because it relates the most important intrinsic and extrinsic quantities at an arbitrary point  $x \in M$ , without any restriction on the dimension/codimension or any further geometric/topological assumptions. This universal inequality was a difficult conjecture in [10, 11], and was finally proved in [12] and [22]. By the suggestion of [6, 24] and the characterization of [12] about the equality case at an arbitrary point, we make the following definition.

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**Definition 1.1.** We denote  $f : M^m \longrightarrow \mathbb{Q}^{m+p}(c)$  a submanifold of dimension  $m$  and codimension  $p$  in a real space form of constant sectional curvature  $c$ . It is a Wintgen ideal submanifold if the equality is attained at every point of  $M^m$  in the DDVV inequality. This happens if, and only if, at every point  $x \in M$  there exists an orthonormal basis  $\{e_1, \dots, e_m\}$  of the tangent plane  $T_x M^m$  and an orthonormal basis  $\{n_1, \dots, n_p\}$  of the normal plane  $T_x^\perp M^m$ , such that the shape operators  $\{A_{n_i}, i = 1, \dots, p\}$  take the form as below [12]:

$$(1.1) \quad A_{n_1} = \begin{pmatrix} \lambda_1 & \mu_0 & 0 & \cdots & 0 \\ \mu_0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix}, A_{n_2} = \begin{pmatrix} \lambda_2 + \mu_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu_0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix},$$

$$A_{n_3} = \lambda_3 I_m, \quad A_{n_r} = 0, r \geq 4.$$

Wintgen ideal submanifolds are abundant. Wintgen first proved the DDVV inequality for surfaces in  $\mathbb{S}^4$ , and characterized the equality case [26]. More general, a surface  $f : M^2 \longrightarrow \mathbb{Q}_c^{2+p}$  of arbitrary codimension  $p$  is Wintgen ideal exactly when the curvature ellipse is a circle at every point [13], which is also equivalent to the Hopf differential being isotropic. For more examples see [2, 8, 9, 17, 19].

An important observation [11, 9] is that the DDVV inequality as well as the equality case are invariant under Möbius transformations of the ambient space. Thus it is appropriate to put the study of Wintgen ideal submanifolds in the framework of Möbius geometry. It follows that Wintgen ideal submanifolds in the sphere  $\mathbb{S}^{m+p}$  or hyperbolic space  $\mathbb{H}^{m+p}$  are the pre-image of a stereographic projection of Wintgen ideal submanifolds in  $\mathbb{R}^{m+p}$ . For the same reason it is no restriction when we describe them in the Euclidean space.

Since there are still many possible examples of Wintgen ideal submanifolds up to Möbius transformations, it is natural to restrict to the best examples with many symmetries, called *Möbius homogeneous submanifolds*. This means that for  $f : M^m \longrightarrow \mathbb{Q}^{m+p}(c)$  and arbitrary two points  $x_1, x_2 \in M^m$ , there exists a Möbius transformation  $\phi$  of  $\mathbb{Q}^{m+p}(c)$  satisfying  $\phi \circ f(x_1) = f(x_2)$  and  $\phi \circ f(M^m) = f(M^m)$ . Such a submanifold is an orbit of a subgroup in the Möbius transformation group.

Our goal in this paper is to classify Möbius homogeneous Wintgen ideal submanifolds of dimension  $m \geq 3$  in  $\mathbb{R}^{m+p}$ . Below are some examples.

**Example 1.2.** Let  $f : S^2 \rightarrow S^{2m}(m \geq 2)$  be one of the Veronese surfaces mentioned at the beginning. As is well-known, such examples are totally isotropic and homogeneous with respect to the isometry group of  $S^{2m}$ . So they are Möbius homogeneous Wintgen ideal submanifolds. They come from the irreducible orthogonal representations of  $SO(3)$ .

**Example 1.3.** Let  $f : \mathbb{R}^2 \rightarrow S^{2m-1}$  be a Clifford-type surface. That means it is homogeneous, flat, and minimal. It comes from a subgroup of the maximal torus group  $T^m \rightarrow SO(2m)$ . Following [3], it is given by  $f = (f^1, \dots, f^{2m})$

$$(1.2) \quad \begin{aligned} f^{2k-1}(x, y) &= r_k \cos(x \cos \theta_k + y \sin \theta_k), \\ f^{2k}(x, y) &= r_k \sin(x \cos \theta_k + y \sin \theta_k), \quad (1 \leq k \leq m) \end{aligned}$$

where  $(r_k, \theta_k)$  are  $m$  real numbers satisfying  $r_k > 0$ ,  $\{\theta_k\}$  are distinctive modulo  $k\pi$ , and

$$r_1^2 + \cdots + r_m^2 = 1, \quad e^{2i\theta_1} r_1^2 + \cdots + e^{2i\theta_m} r_m^2 = 0.$$

Clearly, the flat minimal surface  $f : \mathbb{R}^2 \rightarrow S^{2m-1}$  is an orbit of a 2-dimensional abelian subgroup of  $SO(2m)$ . By direct computation we know that  $f : \mathbb{R}^2 \rightarrow S^{2m-1}$  is Wintgen ideal if, and only if,

$$e^{4i\theta_1} r_1^2 + \dots + e^{4i\theta_m} r_m^2 = 0.$$

**Example 1.4.** Let  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^m$  be a Veronese 2-sphere [1]. Let  $\pi : S^{2m+1} \rightarrow \mathbb{CP}^m$  be the projection map of the Hopf bundle. Then  $\pi^{-1} \circ f : \mathbb{CP}^1 \rightarrow S^{2m+1}$  is a Möbius homogeneous Wintgen ideal submanifold [10].

It comes from the irreducible unitary representations of  $SU(2)$ . When  $m$  is an even number, this submanifold factors as an embedded  $SO(3) = \mathbb{RP}^3$ ; otherwise it is an embedded  $SU(2) = S^3$ .

**Example 1.5.** The cone over an immersed submanifold  $u : M^r \rightarrow S^{r+p} \subset \mathbb{R}^{r+p+1}$  is

$$\begin{aligned} f : \mathbb{R}^+ \times \mathbb{R}^{m-r-1} \times M^r &\rightarrow \mathbb{R}^{m+p}, \\ f(t, y, u) &= (y, tu), \end{aligned}$$

It is a Wintgen ideal submanifold if (and only if)  $u$  is a minimal Wintgen ideal submanifold in  $S^{r+p}$ . (See Section 4 for the proof.)

Our main theorem is as below.

**Theorem 1.6.** Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. Then locally  $f$  is Möbius equivalent to

- (i) a cone over a Veronese surface in  $S^{2k}$ ,
- (ii) a cone over a Clifford-type surface in  $S^{2k+1}$ ,
- (iii) or a cone over  $\pi^{-1} \circ f : \mathbb{CP}^1 \rightarrow S^{2k-1}$ ,
- (iv) or an affine subspace in  $\mathbb{R}^{m+p}$ .

Our conclusions do not extend to Möbius homogeneous Wintgen ideal surfaces, i.e., when  $m = 2$ . In  $S^4$ , any of them is Möbius equivalent to part of the Veronese surface. This follows from the classification of Willmore surfaces with constant Möbius curvature [23], or from an unpublished old manuscript by H. Li, F. Wu and the third author which gave a classification of all Möbius homogeneous surfaces in  $S^4$ . On the other hand, we can modify Example 1.3 to obtain homogeneous, Wintgen ideal (i.e., isotropic), isometric immersions  $\mathbb{R}^2$  in  $S^5$  which are not minimal. Whether there exist other kinds of examples are still unknown to us.

In the rest part of this introduction, we give an overview of the proof and the whole structure of this paper.

We start by reviewing the submanifold theory in Möbius geometry in Section 2. In Section 3 we restrict to Wintgen ideal submanifolds. Due to the specific, simple structure of the (Möbius) second fundamental form, we derive the explicit expressions of the Möbius invariants.

From the statement of the main theorem one can see the importance of the construction by cones (Example 1.5). This is described in detail in Section 4. In particular, we show that the cone  $f$  is Wintgen ideal if and only if the original submanifold  $u$  in the sphere is minimal and Wintgen ideal.

In Section 5, we start to utilize the assumption of Möbius homogeneity. The first structure result is that when the dimension  $m \geq 3$ , the Möbius form  $\Phi$  of a Möbius homogeneous

Wintgen ideal submanifold vanishes. This is proved by contradiction and detailed analysis of the Möbius invariants using the integrable equations.

One crucial ingredient in the discussions of Section 5 and 6 is to re-choose the tangent and normal frames so that the normal connection takes an elegant form. This is the main content of Lemma 5.3, Proposition 6.1 and 6.2.

In Section 7 we prove a somewhat surprising reduction result, namely, if the dimension  $m \geq 3$ , the Möbius homogeneous Wintgen ideal submanifold is a cone over a surface or a three dimensional submanifold in  $\mathbb{S}^{m+p}$ .

In Section 8, we give the proof of our classification theorem. In particular, the only three dimensional Möbius homogeneous and Wintgen ideal examples which are not cones over surfaces must be given by the Hopf bundle over the Veronese surfaces in  $\mathbb{C}P^n$ .

## 2 Submanifolds theory in Möbius geometry

In this section we briefly review the theory of submanifolds in Möbius geometry. For details we refer to [25] and [18].

Let  $\mathbb{R}_1^{m+p+2}$  be the Lorentz space with inner product  $\langle \cdot, \cdot \rangle$  defined by

$$\langle Y, Z \rangle = -Y_0 Z_0 + Y_1 Z_1 + \cdots + Y_{m+p+1} Z_{m+p+1},$$

where  $Y = (Y_0, Y_1, \dots, Y_{m+p+1})$ ,  $Z = (Z_0, Z_1, \dots, Z_{m+p+1}) \in \mathbb{R}^{m+p+2}$ .

Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  be a submanifold without umbilics and assume that  $\{e_i\}$  is an orthonormal basis with respect to the induced metric  $I = df \cdot df$  with  $\{\theta_i\}$  the dual basis. Let  $\{n_\alpha | 1 \leq \alpha \leq p\}$  be a local orthonormal basis for the normal bundle. As usual we denote the second fundamental form and the mean curvature of  $f$  as

$$II = \sum_{ij, \alpha} h_{ij}^\alpha \theta_i \otimes \theta_j n_\alpha, \quad H = \frac{1}{m} \sum_{j, \alpha} h_{jj}^\alpha n_\alpha = \sum_{\alpha} H^\alpha n_\alpha.$$

We define the Möbius position vector  $Y : M^m \rightarrow \mathbb{R}_1^{m+p+2}$  of  $f$  by

$$Y = \rho \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right), \quad \rho^2 = \frac{m}{m-1} \left| II - \frac{1}{m} \text{tr}(II) I \right|^2.$$

It is known that  $Y$  is a well-defined canonical lift of  $f$ . Two submanifolds  $f, \bar{f} : M^m \rightarrow \mathbb{R}^{m+p}$  are Möbius equivalent if there exists  $T$  in the Lorentz group  $\mathbb{O}(m+p+1, 1)$  in  $\mathbb{R}_1^{m+p+2}$  such that  $\bar{Y} = YT$ . It follows immediately that

$$g = \langle dY, dY \rangle = \rho^2 df \cdot df$$

is a Möbius invariant, called the Möbius metric of  $f$ .

Let  $\Delta$  be the Laplacian with respect to  $g$ . Define

$$N = -\frac{1}{m} \Delta Y - \frac{1}{2m^2} \langle \Delta Y, \Delta Y \rangle Y,$$

which satisfies

$$\langle Y, Y \rangle = 0 = \langle N, N \rangle, \quad \langle N, Y \rangle = 1.$$

Let  $\{E_1, \dots, E_m\}$  be a local orthonormal basis for  $(M^m, g)$  with dual basis  $\{\omega_1, \dots, \omega_m\}$ . Write  $Y_j = E_j(Y)$ . Then we have

$$\langle Y_j, Y \rangle = \langle Y_j, N \rangle = 0, \quad \langle Y_j, Y_k \rangle = \delta_{jk}, \quad 1 \leq j, k \leq m.$$

We define

$$\xi_\alpha = H^\alpha \left( \frac{1 + |f|^2}{2}, \frac{1 - |f|^2}{2}, f \right) + (f \cdot n_\alpha, -f \cdot n_\alpha, n_\alpha).$$

Then  $\{\xi_1, \dots, \xi_p\}$  be the orthonormal basis of the orthogonal complement of  $\text{Span}\{Y, N, Y_j | 1 \leq j \leq m\}$ . And  $\{Y, N, Y_j, \xi_\alpha\}$  form a moving frame in  $R_1^{m+p+2}$  along  $M^m$ .

**Remark 2.1.** Geometrically,  $\xi_\alpha$  corresponds to the unique sphere tangent to  $M^m$  at one point  $x$  with normal vector  $n_\alpha$  and the same mean curvature  $H^\alpha(x)$ . We call  $\{\xi_\alpha\}$  the mean curvature spheres of  $M^m$ .

We will use the following range of indices in this section:  $1 \leq i, j, k \leq m; 1 \leq \alpha, \beta \leq p$ . We can write the structure equations as below:

$$\begin{aligned} dY &= \sum_i \omega_i Y_i, \\ dN &= \sum_{ij} A_{ij} \omega_i Y_j + \sum_{i,\alpha} C_i^\alpha \omega_i \xi_\alpha, \\ dY_i &= - \sum_j A_{ij} \omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_{j,\alpha} B_{ij}^\alpha \omega_j \xi_\alpha, \\ d\xi_\alpha &= - \sum_i C_i^\alpha \omega_i Y - \sum_{ij} \omega_i B_{ij}^\alpha Y_j + \sum_\beta \theta_{\alpha\beta} \xi_\beta, \end{aligned}$$

where  $\omega_{ij}$  are the connection 1-forms of the Möbius metric  $g$  and  $\theta_{\alpha\beta}$  the normal connection 1-forms. The tensors

$$\mathbf{A} = \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \quad \mathbf{B} = \sum_{ij\alpha} B_{ij}^\alpha \omega_i \otimes \omega_j \xi_\alpha, \quad \Phi = \sum_{j\alpha} C_j^\alpha \omega_j \xi_\alpha$$

are called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of  $x$ , respectively. The covariant derivatives of  $C_i^\alpha, A_{ij}, B_{ij}^\alpha$  are defined by

$$\begin{aligned} \sum_j C_{i,j}^\alpha \omega_j &= dC_i^\alpha + \sum_j C_j^\alpha \omega_{ji} + \sum_\beta C_j^\beta \theta_{\beta\alpha}, \\ \sum_k A_{ij,k} \omega_k &= dA_{ij} + \sum_k A_{ik} \omega_{kj} + \sum_k A_{kj} \omega_{ki}, \\ \sum_k B_{ij,k}^\alpha \omega_k &= dB_{ij}^\alpha + \sum_k B_{ik}^\alpha \omega_{kj} + \sum_k B_{kj}^\alpha \omega_{ki} + \sum_\beta B_{ij}^\beta \theta_{\beta\alpha}. \end{aligned}$$

The integrability conditions for the structure equations are given by

$$(2.3) \quad A_{ij,k} - A_{ik,j} = \sum_\alpha (B_{ik}^\alpha C_j^\alpha - B_{ij}^\alpha C_k^\alpha),$$

$$(2.4) \quad C_{i,j}^\alpha - C_{j,i}^\alpha = \sum_k (B_{ik}^\alpha A_{kj} - B_{jk}^\alpha A_{ki}),$$

$$(2.5) \quad B_{ij,k}^\alpha - B_{ik,j}^\alpha = \delta_{ij} C_k^\alpha - \delta_{ik} C_j^\alpha,$$

$$(2.6) \quad R_{ijkl} = \sum_\alpha (B_{ik}^\alpha B_{jl}^\alpha - B_{il}^\alpha B_{jk}^\alpha) + \delta_{ik} A_{jl} + \delta_{jl} A_{ik} - \delta_{il} A_{jk} - \delta_{jk} A_{il},$$

$$(2.7) \quad R_{\alpha\beta ij}^\perp = \sum_k (B_{ik}^\alpha B_{kj}^\beta - B_{ik}^\beta B_{kj}^\alpha).$$

Here  $R_{ijkl}$  denote the curvature tensor of  $g$ . Other restrictions on tensors  $\mathbf{A}, \mathbf{B}$  are

$$(2.8) \quad \sum_j B_{jj}^\alpha = 0, \quad \sum_{ijr} (B_{ij}^\alpha)^2 = \frac{m-1}{m},$$

$$(2.9) \quad \text{tr} \mathbf{A} = \sum_j A_{jj} = \frac{1}{2m}(1 + m^2 \kappa).$$

Where  $\kappa = \frac{1}{n(n-1)} \sum_{ij} R_{ijij}$  is its normalized Möbius scalar curvature. We know that all coefficients in the structure equations are determined by  $\{g, \mathbf{B}\}$  and the normal connection  $\{\theta_{\alpha\beta}\}$ . Coefficients of Möbius invariants and the isometric invariants are also related by [25]

$$(2.10) \quad B_{ij}^\alpha = \rho^{-1}(h_{ij}^\alpha - H^\alpha \delta_{ij}),$$

$$(2.11) \quad C_i^\alpha = -\rho^{-2}[H_{,i}^\alpha + \sum_j (h_{ij}^\alpha - H^\alpha \delta_{ij})e_j(\ln \rho)].$$

### 3 Möbius invariants on Wintgen ideal submanifolds

A submanifold  $f : M^m \rightarrow \mathbb{R}^{m+p}$  is a Wintgen ideal submanifold if and only if, at each point of  $M^m$ , there is a suitable frame such that the second fundamental form has the form (1.1). If  $\mu_0 = 0$  in (1.1), then the Wintgen ideal submanifold is totally umbilical submanifold. Next we consider non-umbilical Wintgen ideal submanifolds, that is  $\mu_0 \neq 0$  on  $M^m$  and  $m \geq 3$ .

Since  $\mu_0 \neq 0$ , we can choose a local orthonormal basis  $\{E_1, \dots, E_m\}$  of  $TM^m$  with respect to the Möbius metric  $g$  and a local orthonormal basis  $\{\xi_1, \dots, \xi_p\}$  of  $T^\perp M^m$ , such that the coefficients of the Möbius second fundamental form  $\mathbf{B}$  have the form

$$(3.12) \quad B^1 = \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}; \quad B^\alpha = 0, \quad \alpha \geq 3.$$

By (2.8), the norm of  $\mathbf{B}$  is constant and  $\mu = \sqrt{\frac{m-1}{4m}}$ . Clearly the distribution  $\mathbb{D} = \text{span}\{E_1, E_2\}$  is well-defined. For convenience we adopt the convention below on the range of indices:

$$1 \leq i, j, k, l \leq m, \quad 3 \leq a, b, c \leq m, \quad 1 \leq \alpha, \beta, \gamma \leq p.$$

First we compute the covariant derivatives of  $B_{ij}^\alpha$ . Since the Möbius second fundamental form  $\mathbf{B}$  has the form (3.12), using the definition of the covariant derivatives of  $B_{ij}^\alpha$ , we have

$$(3.13) \quad B_{ab,k}^\delta = 0, \quad 1 \leq \delta \leq p, 1 \leq k \leq m; \quad B_{1a,i}^\alpha = 0, \quad B_{2a,i}^\alpha = 0, \quad \alpha \geq 3,$$

$$\theta_{1\alpha} = \frac{B_{12,1}^\alpha}{\mu} \omega_1 + \frac{B_{12,2}^\alpha}{\mu} \omega_2, \quad \theta_{2\alpha} = \frac{B_{11,1}^\alpha}{\mu} \omega_1 + \frac{B_{11,2}^\alpha}{\mu} \omega_2, \quad \alpha \geq 3.$$

$$(3.14) \quad \omega_{2a} = \sum_i \frac{B_{1a,i}^1}{\mu} \omega_i = - \sum_i \frac{B_{2a,i}^2}{\mu} \omega_i, \quad \omega_{1a} = \sum_i \frac{B_{2a,i}^1}{\mu} \omega_i = \sum_i \frac{B_{1a,i}^2}{\mu} \omega_i.$$

$$(3.15) \quad \begin{aligned} 2\omega_{12} + \theta_{12} &= \sum_i \frac{-B_{11,i}^1}{\mu} \omega_i = \sum_i \frac{B_{22,i}^1}{\mu} \omega_i = \sum_i \frac{B_{12,i}^2}{\mu} \omega_i, \\ B_{12,i}^1 &= 0, \quad B_{11,i}^2 = B_{22,i}^2 = 0. \end{aligned}$$

It follows from (2.5) and (3.13) that, when  $\alpha \geq 3$ ,

$$\begin{aligned} C_1^\alpha &= B_{aa,1}^\alpha - B_{a1,a}^\alpha = 0, \quad C_2^\alpha = B_{aa,2}^\alpha - B_{a2,a}^\alpha = 0; \\ C_a^\alpha &= B_{11,a}^\alpha - B_{1a,1}^\alpha = B_{11,a}^\alpha, \quad C_a^\alpha = B_{22,a}^\alpha - B_{2a,2}^\alpha = B_{22,a}^\alpha. \end{aligned}$$

Since  $\sum_i B_{ii,k}^\delta = 0, 1 \leq \delta \leq p, 1 \leq k \leq m$ , we have

$$C_i^\alpha = 0, \quad \alpha \geq 3.$$

From (3.14) and (3.15), we obtain

$$B_{2a,2}^1 = B_{22,a}^1 = B_{1a,2}^2, \quad B_{1a,1}^2 = 0, \quad B_{2a,2}^2 = 0.$$

This implies that  $C_a^1 = B_{22,a}^1 - B_{2a,2}^1 = 0$ . Similarly  $C_a^2 = 0$ .

The other coefficients of  $\{C_j^r\}$  are obtained similarly as below:

$$(3.16) \quad \begin{aligned} C_1^1 &= -B_{1a,a}^1 = -\mu\omega_{2a}(e_a), \quad C_2^2 = -B_{2a,a}^2 = \mu\omega_{2a}(e_a), \\ C_2^1 &= -B_{2a,a}^1 = -\mu\omega_{1a}(e_a), \quad C_1^2 = -B_{1a,a}^2 = -\mu\omega_{1a}(e_a). \end{aligned}$$

In particular we have

$$(3.17) \quad C_1^1 = -C_2^2, \quad C_2^1 = C_1^2.$$

**Lemma 3.1.** *In the sub-bundles  $\text{Span}\{E_1, E_2\}$  and  $\text{Span}\{\xi_1, \xi_2\}$ , we can always choose new orthonormal basis  $\{E_1, E_2\}$  and  $\{\xi_1, \xi_2\}$  such that the Möbius second fundamental form  $\mathbf{B}$  still takes the form (3.12), and the coefficients of the Möbius form satisfy*

$$C_1^1 = -C_2^2, \quad C_2^1 = C_1^2 = 0, \quad C_a^1 = C_a^2 = 0, \quad C_i^\alpha = 0, \quad \alpha \geq 3.$$

*Proof.* Under a new basis given as below:

$$\begin{cases} \tilde{E}_1 = \cos \theta E_1 + \sin \theta E_2, \\ \tilde{E}_2 = -\sin \theta E_1 + \cos \theta E_2, \end{cases} \quad \begin{cases} \tilde{\xi}_1 = \cos \varphi \xi_1 + \sin \varphi \xi_2, \\ \tilde{\xi}_2 = -\sin \varphi \xi_1 + \cos \varphi \xi_2, \end{cases}$$

we have

$$\begin{aligned} \begin{pmatrix} \tilde{B}_{11}^1 & \tilde{B}_{12}^1 \\ \tilde{B}_{21}^1 & \tilde{B}_{22}^1 \end{pmatrix} &= \begin{pmatrix} \sin(2\theta + \varphi)\mu & \cos(2\theta + \varphi)\mu \\ \cos(2\theta + \varphi)\mu & -\sin(2\theta + \varphi)\mu \end{pmatrix}, \\ \begin{pmatrix} \tilde{B}_{11}^2 & \tilde{B}_{12}^2 \\ \tilde{B}_{21}^2 & \tilde{B}_{22}^2 \end{pmatrix} &= \begin{pmatrix} \cos(2\theta + \varphi)\mu & -\sin(2\theta + \varphi)\mu \\ -\sin(2\theta + \varphi)\mu & -\cos(2\theta + \varphi)\mu \end{pmatrix}, \\ \begin{pmatrix} \tilde{C}_1^1 & \tilde{C}_2^1 \\ \tilde{C}_1^2 & \tilde{C}_2^2 \end{pmatrix} &= \begin{pmatrix} \cos(\theta + \varphi)C_1^1 + \sin(\theta + \varphi)C_2^1 & \cos(\theta + \varphi)C_2^1 - \sin(\theta + \varphi)C_1^1 \\ \cos(\theta + \varphi)C_2^1 - \sin(\theta + \varphi)C_1^1 & -\cos(\theta + \varphi)C_1^1 - \sin(\theta + \varphi)C_2^1 \end{pmatrix}. \end{aligned}$$

Let  $\varphi = -2\theta$ , then the coefficients of the Möbius second fundamental form  $\mathbf{B}$  satisfy (3.12). Clearly there exists a value  $\theta$  such that  $\tilde{C}_2^1 = \tilde{C}_1^2 = 0$ .  $\square$

**Lemma 3.2.** *We can choose a local orthonormal basis  $\{E_3, \dots, E_m\}$  such that*

$$B_{11,4}^1 = B_{11,5}^1 = \dots = B_{11,m}^1 = 0.$$

*Proof.* Let  $E = \sum_{a=3}^m B_{11,a}^1 E_a$ . If  $E = 0$ , then the Lemma is true. If  $E \neq 0$ , then we can choose a local orthonormal basis  $\{\tilde{E}_3, \dots, \tilde{E}_m\}$  in  $\text{Span}\{E_3, \dots, E_m\}$  such that  $\tilde{E}_3 = \frac{E}{|E|}$ . Clearly, under this basis  $B_{11,4}^1 = \dots = B_{11,m}^1 = 0$  as desired.  $\square$

From (3.13), (3.14), (3.15), Lemma 3.1 and Lemma 3.2, we write out the connection forms with respect to the local orthonormal basis  $\{E_1, E_2, \dots, E_m\}$ :

$$\begin{aligned}
 2\omega_{12} + \theta_{12} &= \frac{C_1^1}{\mu} \omega_1 - \frac{B_{11,3}^1}{\mu} \omega_3, \\
 \omega_{13} &= -\frac{B_{11,3}^1}{\mu} \omega_2, \quad \omega_{1i} = 0, i \geq 4, \\
 \omega_{23} &= \frac{B_{11,3}^1}{\mu} \omega_1 - \frac{C_1^1}{\mu} \omega_3, \quad \omega_{2i} = -\frac{C_1^1}{\mu} \omega_i, i \geq 4.
 \end{aligned}
 \tag{3.18}$$

Combining  $C_2^1 = 0$  and the definition of  $C_{i,j}^\alpha$ , we have

$$C_1^1(\omega_{12} + \theta_{12}) = \sum_k C_{2,k}^1 \omega_k.$$

Combining (3.18), we obtain that

$$C_1^1 \omega_{12} = \frac{(C_1^1)^2}{\mu} \omega_1 - \frac{C_1^1 B_{11,3}^1}{\mu} \omega_3 - \sum_k C_{2,k}^1 \omega_k. \tag{3.19}$$

Using  $d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l$ , (3.18) and (3.19), we have the following equations

$$\begin{aligned}
 \sum_{k < l} R_{13kl} \omega_k \wedge \omega_l &= \frac{dB_{11,3}^1}{\mu} \wedge \omega_2 + \sum_k \frac{C_{2,k}^1}{\mu} \omega_k \wedge \omega_3 \\
 &\quad + \frac{(B_{11,3}^1)^2 - (C_1^1)^2}{\mu^2} \omega_1 \wedge \omega_3, \\
 \sum_{k < l} R_{1ikl} \omega_k \wedge \omega_l &= \frac{-(C_1^1)^2}{\mu^2} \omega_1 \wedge \omega_i + \frac{C_1^1 B_{11,3}^1}{\mu^2} \omega_3 \wedge \omega_i \\
 &\quad + \sum_k \frac{C_{2,k}^1}{\mu} \omega_k \wedge \omega_i - \frac{B_{11,3}^1}{\mu} \omega_2 \wedge \omega_{3i}, \quad i \geq 4,
 \end{aligned}
 \tag{3.20}$$

$$\begin{aligned}
 \sum_{k < l} R_{23kl} \omega_k \wedge \omega_l &= \frac{-dB_{11,3}^1}{\mu} \wedge \omega_1 + \sum_k \frac{C_{1,k}^1}{\mu} \omega_k \wedge \omega_3 \\
 &\quad + \frac{(B_{11,3}^1)^2 - (C_1^1)^2}{\mu^2} \omega_2 \wedge \omega_3 - \frac{2C_1^1 B_{11,3}^1}{\mu^2} \omega_1 \wedge \omega_2, \\
 \sum_{k < l} R_{2ikl} \omega_k \wedge \omega_l &= \frac{-(C_1^1)^2}{\mu^2} \omega_2 \wedge \omega_i + \sum_k \frac{C_{1,k}^1}{\mu} \omega_k \wedge \omega_i \\
 &\quad + \frac{B_{11,3}^1}{\mu} \omega_1 \wedge \omega_{3i}, \quad i \geq 4.
 \end{aligned}
 \tag{3.21}$$



Using (2.6), from (3.20) and (3.21), we obtain that the coefficients of the Blaschke tensor satisfy

$$(3.22) \quad \begin{aligned} A_{11} + A_{33} &= \frac{C_{2,1}^1}{\mu} + \frac{(B_{11,3}^1)^2 - (C_1^1)^2}{\mu^2}; \quad A_{11} + A_{ii} = \frac{C_{2,1}^1}{\mu} - \frac{(C_1^1)^2}{\mu^2}, i \geq 4; \\ A_{22} + A_{33} &= \frac{C_{1,2}^1}{\mu} + \frac{(B_{11,3}^1)^2 - (C_1^1)^2}{\mu^2}; \quad A_{22} + A_{ii} = \frac{C_{1,2}^1}{\mu} - \frac{(C_1^1)^2}{\mu^2}, i \geq 4. \end{aligned}$$

$$(3.23) \quad \begin{aligned} A_{23} &= \frac{C_{1,3}^1}{\mu} = \frac{E_1(B_{11,3}^1)}{\mu}, \\ A_{13} &= \frac{2C_1^1 B_{11,3}^1}{\mu^2} - \frac{E_2(B_{11,3}^1)}{\mu} = \frac{C_{2,3}^1}{\mu} + \frac{C_1^1 B_{11,3}^1}{\mu^2}, \\ A_{1i} &= \frac{C_{2,i}^1}{\mu} = -\frac{B_{11,3}^1}{\mu} \omega_{3i}(E_2), \quad A_{2i} = \frac{C_{1,i}^1}{\mu} = \frac{B_{11,3}^1}{\mu} \omega_{3i}(E_1), i \geq 4, \end{aligned}$$

$$(3.24) \quad \begin{aligned} A_{ij} &= 0, \quad i, j \geq 4, \quad i \neq j, \\ A_{12} &= \frac{C_{2,2}^1}{\mu} - \frac{E_3(B_{11,3}^1)}{\mu} = \frac{C_{1,1}^1}{\mu} + \frac{E_3(B_{11,3}^1)}{\mu}. \end{aligned}$$

$$(3.25) \quad \begin{aligned} E_i(B_{11,3}^1) &= 0, \quad E_3(B_{11,3}^1) = B_{11,3}^1 \omega_{3i}(E_i), \quad i \geq 4, \\ \omega_{3i}(E_3) &= 0, i \geq 4, \quad \omega_{3i}(E_j) = 0, \quad i, j \geq 4, i \neq j. \end{aligned}$$

From (3.22), we obtain the coefficients of the Blaschke tensor satisfy

$$A_{ij} = A_4 \delta_{ij}, \quad i, j \geq 4, \quad A_4 \triangleq A_{44}.$$

## 4 Wintgen ideal submanifolds constructed by cones

**Definition 4.1.** Let  $u : M^r \longrightarrow \mathbb{S}^{r+p} \subset \mathbb{R}^{r+p+1}$  be an immersed submanifold. We define the cone over  $u$  in  $\mathbb{R}^{m+p}$  as

$$\begin{aligned} f : R^+ \times \mathbb{R}^{m-r-1} \times M^r &\longrightarrow \mathbb{R}^{m+p}, \\ f(t, y, u) &= (y, tu), \end{aligned}$$

**Proposition 4.2.** Let  $u : M^r \longrightarrow \mathbb{S}^{r+p}$  be an immersed submanifold. Then the cone  $f = (y, tu) : R^+ \times \mathbb{R}^{m-r-1} \times M^r \longrightarrow \mathbb{R}^{m+p}$  is a Wintgen ideal submanifold if and only if  $u$  is a minimal Wintgen ideal submanifold in  $\mathbb{S}^{r+p}$ .

*Proof.* The first and second fundamental forms of  $f$  are, respectively,

$$(4.26) \quad I = t^2 I_u + I_{\mathbb{R}^{m-r}}, \quad II = t II_u,$$

where  $I_u, II_u$  are the first and second fundamental forms of  $u$ , respectively, and  $I_{\mathbb{R}^{m-r}}$  denotes the standard metric of  $\mathbb{R}^{m-r}$ . The conclusion follows easily.  $\square$

The Möbius position vector  $Y : R^+ \times \mathbb{R}^{m-r-1} \times M^r \longrightarrow \mathbb{R}_1^{m+p+2}$  of the cone  $f$  is

$$Y = \rho_0 \left( \frac{1+t^2+|y|^2}{2t}, \frac{1-t^2-|y|^2}{2t}, y, u \right),$$

where  $\rho_0^2 = \frac{m}{m-1}(|II_u|^2 - mH_u^2) : M^r \longrightarrow \mathbb{R}$ , and  $y : \mathbb{R}^{m-r-1} \longrightarrow \mathbb{R}^{m-r-1}$  is the identity map. Let

$$\mathbb{H}^{m-r} = \{(y_0, y) \in \mathbb{R}^{m-r+1} : -y_0^2 + |y|^2 = -1, y_0 \geq 1\} \cong R^+ \times \mathbb{R}^{m-r-1},$$

then  $(\frac{1+t^2+|y|^2}{2t}, \frac{1-t^2-|y|^2}{2t}, y) : R^+ \times \mathbb{R}^{m-r-1} \cong \mathbb{H}^{m-r} \rightarrow \mathbb{H}^{m-r}$  is nothing else but the identity map. And the Möbius position vector of the cone  $f$  is

$$(4.27) \quad Y = \rho_0(id, u) : \mathbb{H}^{m-r} \times M^r \rightarrow \mathbb{H}^{m-r} \times \mathbb{S}^{r+p} \subset \mathbb{R}_1^{m+p+2},$$

where  $\rho_0 \in C^\infty(M^r)$  and  $id : \mathbb{H}^{m-r} \rightarrow \mathbb{H}^{m-r}$  is a identity map.

The möbius metric of the cone  $f$  is

$$g = \rho_0^2(I_u + I_{\mathbb{H}^{m-r}}),$$

where  $I_{\mathbb{H}^{m-r}}$  is the standard hyperbolic metric of  $\mathbb{H}^{m-r}$ .

From (4.27) we have the following result.

**Proposition 4.3.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  be an immersed submanifold without umbilical points. If there exists a submanifold  $u : M^r \rightarrow \mathbb{S}^{r+p}$  such that the Möbius position vector of  $f$  is*

$$Y = \rho_0(id, u) : \mathbb{H}^{m-r} \times M^r \rightarrow \mathbb{H}^{m-r} \times \mathbb{S}^{r+p} \subset \mathbb{R}_1^{m+p+2},$$

where  $\rho_0 \in C^\infty(M^r)$  and  $id : \mathbb{H}^{m-r} \rightarrow \mathbb{H}^{m-r}$  is the identity map. Then  $f$  is a cone over  $u$ .

By computation, and combining with (2.11), (4.26), we have the following result.

**Proposition 4.4.** *Let the cone  $f = (y, tu) : R^+ \times \mathbb{R}^{m-r-1} \times M^r \rightarrow \mathbb{R}^{m+p}$  be a Wintgen ideal submanifold. Then the Möbius form  $\Phi$  of  $f$  vanishes if and only if the Möbius form  $\Phi_u$  of  $u : M^r \rightarrow \mathbb{S}^{r+p}$  vanishes.*

## 5 The vanishing of the Möbius form

A common feature of all the examples of Möbius homogeneous, Wintgen ideal submanifolds described in the introduction is that they all have vanishing Möbius form, i.e.,  $\Phi = 0$ . For Examples 1.2 and 1.3, this follows from Proposition 4.4 and the results in [14]. For Example 1.4 in  $S^5$ , this has been verified in [27].

Conversely, in this section we show that any Möbius homogeneous, Wintgen ideal submanifold must have this property. This will be shown by contradiction.

Note that starting from this section, the assumption of Möbius homogeneity will be used, whose basic consequence is that any Möbius invariant geometric quantity as a function well-defined at every point of the underlying manifold must be a constant.

**Lemma 5.1.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$ , ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $\Phi \neq 0$ , then  $B_{11,3}^1 = 0$ .*

*Proof.* By assumption  $\Phi \neq 0$ , i.e.,  $C_1^1 \neq 0$ . By Lemma 3.1, the normal vector field  $\xi_1$  and tangent vector fields  $\{E_1, E_2\}$  are determined up to a sign. Since  $f$  is Möbius homogeneous, the function  $C_1^1$  takes value as a constant. Similarly, if  $B_{11,3}^1 \neq 0$ , then the function  $B_{11,3}^1$  is well-defined function, hence a non-zero constant.

From  $dC_1^1 = \sum_k C_{1,k}^1 \omega_k$ , we have

$$C_{1,k}^1 = 0, \quad 1 \leq k \leq m.$$

From (3.23), (3.24) and (3.25), we obtain

$$(5.28) \quad A_{21} = A_{23} = A_{24} = \cdots = A_{2m} = 0, \quad A_{13} = \frac{2C_1^1 B_{11,3}^1}{\mu^2}, \quad C_{2,3}^1 = \frac{C_1^1 B_{11,3}^1}{\mu}.$$

If  $B_{11,3}^1 \neq 0$ , from (3.23) and (3.25), we have

$$\omega_{3i} = -\frac{\mu A_{1i}}{B_{11,3}^1} \omega_2, \quad i \geq 4.$$

Using  $d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l$ , there follows

$$(5.29) \quad \begin{aligned} \sum_{k < l} R_{3ikl} \omega_k \wedge \omega_l &= d\left(\frac{\mu A_{1i}}{B_{11,3}^1}\right) \wedge \omega_2 - \omega_2 \wedge \left(\sum_{k \geq 4} \frac{\mu A_{1m}}{B_{11,3}^1} \omega_{mi}\right) + \frac{C_1^1 B_{11,3}^1}{\mu^2} \omega_1 \wedge \omega_i \\ &- \frac{\mu A_{1i}}{B_{11,3}^1} \left[ \frac{C_{2,2}^1}{C_1^1} \omega_1 \wedge \omega_2 + \frac{C_{2,3}^1}{C_1^1} \omega_1 \wedge \omega_3 + \omega_1 \wedge \left(\sum_{k \geq 4} \frac{C_{2,k}^1}{C_1^1} \omega_k\right) \right] - \frac{(C_1^1)^2}{\mu^2} \omega_3 \wedge \omega_i. \end{aligned}$$

Comparing  $\omega_1 \wedge \omega_i$  in (5.29), there should be

$$R_{3i1i} = -\frac{\mu A_{1i}}{B_{11,3}^1} \frac{C_{2,i}^1}{C_1^1} + \frac{C_1^1 B_{11,3}^1}{\mu^2}.$$

Since  $A_{13} = \frac{2C_1^1 B_{11,3}^1}{\mu^2}$  and  $C_{2,i}^1 = \mu A_{1i}$ , we obtain

$$\frac{(C_1^1)^2 (B_{11,3}^1)^2}{\mu^4} + (A_{1i})^2 = 0,$$

which is a contradiction. So  $B_{11,3}^1 = 0$ .  $\square$

**Lemma 5.2.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$ , ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $\Phi \neq 0$ , then the coefficients of the Blaschke tensor satisfy*

$$\begin{aligned} A_{22} &= A_{33} = \cdots = A_{mm} = -\frac{(C_1^1)^2}{2\mu^2}, \\ (A_{11} - A_{22})^2 &= \frac{(C_1^1)^2}{\mu^2} (A_{11} - A_{22}) + 2(C_1^1)^2, \end{aligned}$$

and curvature tensor satisfies

$$4\mu^2 = R_{1212} + \sum_{\alpha \geq 3} \frac{(B_{11,2}^\alpha)^2 + (B_{22,1}^\alpha)^2}{\mu^2}.$$

*Proof.* From (3.24), (3.25), (5.28) and Lemma 5.1, we obtain

$$(5.30) \quad \begin{aligned} A_{12} &= A_{13} = \cdots = A_{1m} = 0, \quad C_{2,2}^1 = C_{2,3}^1 = \cdots = C_{2,m}^1 = 0, \\ \omega_{12} &= \left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right) \omega_1, \quad \omega_{1i} = 0, \quad \omega_{2i} = -\frac{C_1^1}{\mu} \omega_i, \quad i \geq 3. \end{aligned}$$

Noticing that local functions  $A_{ii}$  ( $1 \leq i \leq m$ ) are constant, from the definition of  $A_{ij,k}$ , (5.28), and (5.30), we get

$$\begin{aligned} A_{12,1} &= (A_{11} - A_{22}) \left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right), \quad A_{11,2} = 0, \\ A_{23,3} &= (A_{33} - A_{22}) \frac{C_1^1}{\mu}, \quad A_{33,2} = 0. \end{aligned}$$

Using (2.3) and (3.22), we have

$$(5.31) \quad \begin{aligned} A_{22} &= A_{33} = \cdots = A_{mm} = -\frac{(C_1^1)^2}{2\mu^2}, \\ (A_{11} - A_{22})^2 &= \frac{(C_1^1)^2}{\mu^2}(A_{11} - A_{22}) + 2(C_1^1)^2. \end{aligned}$$

Using  $d\omega_{12} - \sum_k \omega_{1k} \wedge \omega_{k2} = -\frac{1}{2} \sum_{kl} R_{12kl} \omega_k \wedge \omega_l$  and (5.30), we have

$$(5.32) \quad R_{1212} = -\left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right)^2.$$

From (3.19) and (5.30), we get

$$(5.33) \quad \theta_{12} = \left(\frac{2C_{2,1}^1}{C_1^1} - \frac{C_1^1}{\mu}\right)\omega_1, \quad d\omega_1 = \left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right)\omega_1 \wedge \omega_2, \quad d\omega_2 = 0.$$

Using  $d\theta_{12} - \sum_\tau \theta_{1\tau} \wedge \theta_{\tau 2} = -\frac{1}{2} \sum_{kl} R_{12kl}^\perp \omega_k \wedge \omega_l$  and (3.13), we have

$$(5.34) \quad 4\mu^2 = R_{1212} + \sum_{\alpha \geq 3} \frac{(B_{11,2}^\alpha)^2 + (B_{22,1}^\alpha)^2}{\mu^2}. \quad \square$$

**Lemma 5.3.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$ , ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $\Phi \neq 0$  and  $p \geq 3$ , then we can choose orthonormal frames in  $\text{Span}\{\xi_3, \dots, \xi_p\}$  such that the normal connection have the following form*

$$\begin{aligned} &\left\{ \begin{aligned} &\left\{ \begin{aligned} \theta_{13} &= a_0 \omega_2, \\ \theta_{23} &= -a_0 \omega_1, \end{aligned} \right. &\left\{ \begin{aligned} \theta_{14} &= a_0 \omega_1, \\ \theta_{24} &= a_0 \omega_2, \end{aligned} \right. &\left\{ \begin{aligned} \theta_{1\alpha} &= 0, \quad \alpha \geq 5, \\ \theta_{2\alpha} &= 0, \quad \alpha \geq 5, \end{aligned} \right. \\ \theta_{34} &= \left(3\frac{C_{2,1}^1}{C_1^1} - 2\frac{C_1^1}{\mu}\right)\omega_1, \quad a_0 \neq 0. \end{aligned} \right. \\ &\left\{ \begin{aligned} &\left\{ \begin{aligned} \theta_{(2k+1)(2k+3)} &= a_k \omega_2, \\ \theta_{(2k+2)(2k+3)} &= -a_k \omega_1, \end{aligned} \right. &\left\{ \begin{aligned} \theta_{(2k+1)(2k+4)} &= a_k \omega_1, \\ \theta_{(2k+2)(2k+4)} &= a_k \omega_2, \end{aligned} \right. &\left\{ \begin{aligned} \theta_{(2k+1)\alpha} &= 0, \quad \alpha \geq 2k+5, \\ \theta_{(2k+2)\alpha} &= 0, \quad \alpha \geq 2k+5, \end{aligned} \right. \\ \theta_{(2k+3)(2k+4)} &= [(k+3)\frac{C_{2,1}^1}{C_1^1} - (k+2)\frac{C_1^1}{\mu}]\omega_1, \quad a_k \neq 0. \end{aligned} \right. \\ &\left\{ \begin{aligned} \theta_{(2k+3)\alpha} &= 0, \quad \alpha \geq 2k+5, \\ \theta_{(2k+4)\alpha} &= 0, \quad \alpha \geq 2k+5, \end{aligned} \right. \end{aligned}$$

*Proof.* Since  $p \geq 3$ , from (3.13), we can rechoose an orthonormal basis in  $\text{Span}\{\xi_3, \dots, \xi_p\}$  such that

$$(5.35) \quad \left\{ \begin{aligned} \theta_{13} &= \frac{B_{11,2}^3}{\mu} \omega_1 + \frac{B_{22,1}^3}{\mu} \omega_2, \\ \theta_{23} &= -\frac{B_{22,1}^3}{\mu} \omega_1 + \frac{B_{11,2}^3}{\mu} \omega_2, \end{aligned} \right. \left\{ \begin{aligned} \theta_{14} &= \frac{B_{11,2}^4}{\mu} \omega_1, \\ \theta_{24} &= \frac{B_{11,2}^4}{\mu} \omega_2, \end{aligned} \right. \left\{ \begin{aligned} \theta_{1\alpha} &= 0, \quad \alpha \geq 5, \\ \theta_{2\alpha} &= 0, \quad \alpha \geq 5, \end{aligned} \right.$$

In fact, if  $E = \sum_{\alpha \geq 3} B_{22,1}^\alpha \xi_\alpha \neq 0$ , let  $\tilde{\xi}_3 = \frac{E}{|E|}$ . If  $E = 0$ , let  $\tilde{\xi}_3 = \xi_3$ . If  $\tilde{E} = \sum_{\alpha \geq 4} B_{11,2}^\alpha \xi_\alpha \neq 0$ , let  $\tilde{\xi}_4 = \frac{\tilde{E}}{|\tilde{E}|}$ . If  $\tilde{E} = 0$ , let  $\tilde{\xi}_4 = \xi_4$ . Therefore under the orthonormal basis  $\{\tilde{\xi}_3, \tilde{\xi}_4, \dots, \tilde{\xi}_p\}$ , we have (5.35).

Since  $f$  is Möbius homogeneous and the normal vector fields  $\xi_3, \xi_4$  are determined up to a sign, then  $B_{11,2}^3, B_{22,1}^3$  and  $B_{11,2}^4$  are constants (otherwise they are equal to zero). Noticing

$R_{\alpha\beta kl}^\perp = 0$  for  $\alpha = 1, 2, \beta = 3, 4$ , and using  $d\theta_{\alpha\beta} - \sum_\tau \theta_{\alpha\tau} \wedge \theta_{\tau\beta} = -\frac{1}{2} \sum_{kl} R_{\alpha\beta kl}^\perp \omega_k \wedge \omega_l$  and (5.35), we obtain

$$\begin{aligned}
 (5.36) \quad & \frac{B_{11,2}^3}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 = \frac{B_{11,2}^4}{\mu} \omega_1 \wedge \theta_{34}, \\
 & \frac{B_{22,1}^3}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 = -\frac{B_{11,2}^4}{\mu} \omega_2 \wedge \theta_{34}, \\
 & \left( -\frac{B_{22,1}^3}{\mu} \omega_1 + \frac{B_{11,2}^3}{\mu} \omega_2 \right) \wedge \theta_{34} = 0, \\
 & \frac{B_{11,2}^4}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 = -\left( \frac{B_{11,2}^3}{\mu} \omega_1 + \frac{B_{22,1}^3}{\mu} \omega_2 \right) \wedge \theta_{34}.
 \end{aligned}$$

We eliminate the term  $\theta_{34}$  in (5.36) to obtain

$$(5.37) \quad \frac{B_{22,1}^3}{\mu} \frac{B_{11,2}^3}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) = 0.$$

If  $2 \frac{C_1^1}{\mu} = 3 \frac{C_{2,1}^1}{C_1^1}$ , i.e.,  $\frac{C_{2,1}^1}{\mu} = \frac{2(C_1^1)^2}{3\mu^2}$ , noting  $\frac{C_{2,1}^1}{\mu} = A_{11} - A_{22}$ , then from (5.31) we have

$$\frac{4}{9} \frac{(C_1^1)^4}{\mu^4} = \frac{2(C_1^1)^4}{3\mu^4} + 2(C_1^1)^2,$$

which implies  $-\frac{2}{9} \frac{(C_1^1)^4}{\mu^4} = 2(C_1^1)^2$ , and is a contradiction. So  $2 \frac{C_1^1}{\mu} - 3 \frac{C_{2,1}^1}{C_1^1} \neq 0$ .

If  $B_{22,1}^3 = 0$ , from (5.36), we have

$$\begin{cases} \frac{B_{11,2}^3}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 - \frac{B_{11,2}^4}{\mu} \omega_1 \wedge \theta_{34} = 0, \\ \frac{B_{11,2}^4}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 + \frac{B_{11,2}^3}{\mu} \omega_1 \wedge \theta_{34} = 0, \end{cases}$$

which imply that  $\frac{B_{11,2}^3}{\mu} = \frac{B_{11,2}^4}{\mu} = 0$ . Thus  $\sum_{\alpha \geq 3} \frac{(B_{11,2}^\alpha)^2 + (B_{22,1}^\alpha)^2}{\mu^2} = 0$ , and equation (5.34) implies  $4\mu^2 = R_{1212}$ . Since  $R_{1212} = -\left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right)^2$ , so this is a contradiction. Thus  $B_{22,1}^3 \neq 0$ .

Thus  $B_{11,2}^3 = 0$ . From (5.35) and (5.36) we have

$$\begin{cases} \frac{B_{22,1}^3}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 + \frac{B_{11,2}^4}{\mu} \omega_2 \wedge \theta_{34} = 0, \\ \frac{B_{11,2}^4}{\mu} \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 + \frac{B_{22,1}^3}{\mu} \omega_2 \wedge \theta_{34} = 0, \end{cases}$$

which imply  $\frac{B_{22,1}^3}{\mu} = \pm \frac{B_{11,2}^4}{\mu}$ . We assume  $a_0 \triangleq \frac{B_{22,1}^3}{\mu} = \frac{B_{11,2}^4}{\mu} \neq 0$ , otherwise let  $\tilde{\xi}_4 = -\xi_4$ . From (5.35) and (5.36), we have

$$(5.38) \quad \begin{cases} \theta_{13} = a_0 \omega_2, \\ \theta_{23} = -a_0 \omega_1, \end{cases} \quad \begin{cases} \theta_{14} = a_0 \omega_1, \\ \theta_{24} = a_0 \omega_2, \end{cases} \quad \begin{cases} \theta_{1\alpha} = 0, & \alpha \geq 5, \\ \theta_{2\alpha} = 0, & \alpha \geq 5, \end{cases}$$

and

$$(5.39) \quad \theta_{34} = \left( 3 \frac{C_{2,1}^1}{C_1^1} - 2 \frac{C_1^1}{\mu} \right) \omega_1.$$

Using  $d\theta_{1\alpha} - \sum_{\tau} \theta_{1\tau} \wedge \theta_{\tau\alpha} = -\frac{1}{2} \sum_{kl} R_{1\alpha kl}^{\perp} \omega_k \wedge \omega_l$  and (5.38), noting  $R_{1\alpha kl}^{\perp} = 0, \alpha \geq 5$ , we have

$$-\omega_1 \wedge \theta_{3\alpha} + \omega_2 \wedge \theta_{4\alpha} = 0, \quad \omega_2 \wedge \theta_{3\alpha} + \omega_1 \wedge \theta_{4\alpha} = 0, \quad \alpha \geq 5.$$

Thus we can assume that

$$(5.40) \quad \begin{cases} \theta_{3\alpha} = a_1^{\alpha} \omega_1 + a_2^{\alpha} \omega_2, \\ \theta_{4\alpha} = -a_2^{\alpha} \omega_1 + a_1^{\alpha} \omega_2, \end{cases} \quad \alpha \geq 5.$$

We can choose a new orthonormal frame locally in  $Span\{\xi_5, \dots, \xi_p\}$  such that

$$(5.41) \quad \begin{cases} \theta_{35} = a\omega_1 + b\omega_2, \\ \theta_{45} = -b\omega_1 + a\omega_2, \end{cases} \quad \begin{cases} \theta_{36} = c\omega_1, \\ \theta_{46} = c\omega_2, \end{cases} \quad \begin{cases} \theta_{3\alpha} = 0, & \alpha \geq 7, \\ \theta_{4\alpha} = 0, & \alpha \geq 7, \end{cases}$$

Using  $d\theta_{35} = \sum_{\tau} \theta_{3\tau} \wedge \theta_{\tau 5}$  and (5.39), we have

$$(5.42) \quad a \left( 4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 = c\omega_1 \wedge \theta_{56}.$$

Similarly, we have

$$(5.43) \quad \begin{aligned} b \left( 4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 &= -c\omega_2 \wedge \theta_{56}, \\ c \left( 4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 &= -(a\omega_1 + b\omega_2) \wedge \theta_{56}, \\ -b\omega_1 \wedge \theta_{56} + a\omega_2 \wedge \theta_{56} &= 0. \end{aligned}$$

Using the equations (5.42) and (5.43), we can obtain

$$(5.44) \quad ab \left( 4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \right) = 0.$$

If  $4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} = 0$ , i.e.,  $\frac{C_{2,1}^1}{\mu} = \frac{3(C_1^1)^2}{4\mu^2}$ , noting  $\frac{C_{2,1}^1}{\mu} = A_{11} - A_{22}$ , then from (5.31) we have

$$\frac{9}{16} \frac{(C_1^1)^4}{\mu^4} = \frac{3(C_1^1)^4}{4\mu^4} + 2(C_1^1)^2,$$

which is a contradiction. So  $4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \neq 0$ .

If  $b = 0$ , from (5.42) and (5.43), we have

$$(5.45) \quad \begin{aligned} a \left( 4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 - c\omega_1 \wedge \theta_{56} &= 0, \\ c \left( 4 \frac{C_{2,1}^1}{C_1^1} - 3 \frac{C_1^1}{\mu} \right) \omega_1 \wedge \omega_2 + a\omega_1 \wedge \theta_{56} &= 0. \end{aligned}$$

Which implies  $a = c = 0$ . From (5.38) and (5.41), we have

$$(5.46) \quad d\theta_{34} = \sum_{\tau} \theta_{3\tau} \wedge \theta_{\tau 4} = \theta_{31} \wedge \theta_{14} + \theta_{32} \wedge \theta_{24} = 2a_0^2 \omega_1 \wedge \omega_2.$$

On the other hand, from (5.39), we have

$$d\theta_{34} = \left(3\frac{C_{2,1}^1}{C_1^1} - 2\frac{C_1^1}{\mu}\right) \left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right) \omega_1 \wedge \omega_2.$$

Thus we have

$$(5.47) \quad \left(3\frac{C_{2,1}^1}{C_1^1} - 2\frac{C_1^1}{\mu}\right) \left(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1}\right) = 2a_0^2.$$

Combining with  $C_{2,1}^1 = \mu(A_{11} - A_{22})$  and  $A_{22} = -\frac{(C_1^1)^2}{2\mu^2}$ , we have

$$(5.48) \quad 2a_0^2 = 2A_{11} - \frac{(C_1^1)^2}{\mu^2} - 6\mu^2.$$

From (5.32), we have

$$R_{1212} < 0, \text{ i.e., } A_{11} < 2\mu^2 + \frac{C_1^1}{2\mu^2}.$$

Thus

$$2A_{11} - \frac{(C_1^1)^2}{\mu^2} - 6\mu^2 < -2\mu^2.$$

which is in contradiction with (5.48). Thus  $b \neq 0$ .

Therefore  $a = 0$ , From (5.43), we have

$$(5.49) \quad \begin{aligned} b \left(4\frac{C_{2,1}^1}{C_1^1} - 3\frac{C_1^1}{\mu}\right) \omega_1 \wedge \omega_2 &= -c\omega_2 \wedge \theta_{56}, \\ c \left(4\frac{C_{2,1}^1}{C_1^1} - 3\frac{C_1^1}{\mu}\right) \omega_1 \wedge \omega_2 &= -b\omega_2 \wedge \theta_{56}, \\ \omega_1 \wedge \theta_{56} &= 0, \end{aligned}$$

which implies  $b = \pm c$ , and we can assume that  $b = c = a_1$ . Thus

$$(5.50) \quad \begin{cases} \theta_{35} = a_1\omega_2, \\ \theta_{45} = -a_1\omega_1, \end{cases} \quad \begin{cases} \theta_{36} = a_1\omega_1, \\ \theta_{46} = a_1\omega_2, \end{cases} \quad \begin{cases} \theta_{3\alpha} = 0, & \alpha \geq 7, \\ \theta_{4\alpha} = 0, & \alpha \geq 7, \end{cases}$$

and

$$(5.51) \quad \theta_{56} = \left(4\frac{C_{2,1}^1}{C_1^1} - 3\frac{C_1^1}{\mu}\right) \omega_1.$$

Repeating the process (5.39)–(5.51), we have

$$\begin{aligned} \left\{ \begin{aligned} &\begin{cases} \theta_{(2k+1)(2k+3)} = a_k\omega_2, \\ \theta_{(2k+2)(2k+3)} = -a_k\omega_1, \end{cases} && \begin{cases} \theta_{(2k+1)(2k+4)} = a_k\omega_1, \\ \theta_{(2k+2)(2k+4)} = a_k\omega_2, \end{cases} && \begin{cases} \theta_{(2k+1)\alpha} = 0, & \alpha \geq 2k+5, \\ \theta_{(2k+2)\alpha} = 0, & \alpha \geq 2k+5, \end{cases} \\ &&& \theta_{(2k+3)(2k+4)} = [(k+3)\frac{C_{2,1}^1}{C_1^1} - (k+2)\frac{C_1^1}{\mu}]\omega_1, && a_k \neq 0. \end{aligned} \right. \\ \begin{cases} \theta_{(2k+3)\alpha} = 0, & \alpha \geq 2k+5, \\ \theta_{(2k+4)\alpha} = 0, & \alpha \geq 2k+5, \end{cases} \end{aligned}$$

Thus we finish the proof of Lemma 5.3.  $\square$

**Remark 5.4.** During the proof of Lemma 5.3, we can assume that the codimension of  $f$  is sufficiently large, otherwise we consider  $f : M^m \rightarrow \mathbb{R}^{m+p} \hookrightarrow \mathbb{R}^{m+p+k}$  as a submanifold in  $\mathbb{R}^{m+p+k}$ , which also is a Möbius homogeneous Wintgen ideal submanifold in  $\mathbb{R}^{m+p+k}$ .

**Proposition 5.5.** Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$ , ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold, then the Möbius form vanishes, i.e.,  $\Phi = 0$ .

*Proof.* If  $p = 2$ , then equation (5.34) implies  $4\mu^2 = R_{1212}$ . This is a contradiction since  $R_{1212} = -(\frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1})^2$ . Thus  $\Phi = 0$  and we finish the proof.

If  $p \geq 3$ . Noticing that  $R_{\alpha\beta kl}^\perp = 0$  for  $\alpha = 2k+3, \beta = 2k+4$ , from Lemma 5.3, we have

$$d\theta_{(2k+3)(2k+4)} = \sum_{\tau} \theta_{(2k+3)\tau} \wedge \theta_{\tau(2k+4)} = 2a_k^2 \omega_1 \wedge \omega_2,$$

and

$$d\theta_{(2k+3)(2k+4)} = \left[ (k+3) \frac{C_{2,1}^1}{C_1^1} - (k+2) \frac{C_1^1}{\mu} \right] \left[ \frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1} \right] \omega_1 \wedge \omega_2.$$

Thus we have

$$(5.52) \quad \left[ (k+3) \frac{C_{2,1}^1}{C_1^1} - (k+2) \frac{C_1^1}{\mu} \right] \left[ \frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1} \right] = 2a_k^2.$$

From (5.31) we have

$$\left[ (k+3) \frac{C_{2,1}^1}{C_1^1} - (k+2) \frac{C_1^1}{\mu} \right] \left[ \frac{C_1^1}{\mu} - \frac{C_{2,1}^1}{C_1^1} \right] = (k+2)A_{11} - 2(k+3)\mu^2 - (k+2) \frac{C_1^1}{2\mu^2}.$$

Since  $R_{1212} < 0$ , i.e.,  $A_{11} < 2\mu^2 + \frac{C_1^1}{2\mu^2}$ , Thus

$$(k+2)A_{11} - 2(k+3)\mu^2 - (k+2) \frac{C_1^1}{2\mu^2} < -2\mu^2.$$

The (5.52) implies Thus  $2a_k^2 < -2\mu^2$ , which is a contradiction. Thus  $\Phi = 0$ .  $\square$

Since the Wintgen ideal submanifold  $f$  is Möbius homogeneous, then under the orthonormal basis  $\{E_1, \dots, E_m\}$ ,

$$(5.53) \quad \begin{aligned} 2\omega_{12} + \theta_{12} &= -\frac{B_{11,3}^1}{\mu} \omega_3, \\ \omega_{13} &= -\frac{B_{11,3}^1}{\mu} \omega_2, \quad \omega_{1i} = 0, i \geq 4, \\ \omega_{23} &= \frac{B_{11,3}^1}{\mu} \omega_1, \quad \omega_{2i} = 0, i \geq 4. \end{aligned}$$

From (3.22), we can obtain the following result,

**Proposition 5.6.** Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. Then we can choose the orthonormal basis  $\{E_1, \dots, E_m\}$  such that

$$(A_{ij}) = \text{diag}(A_1, A_1, -A_1, \dots, -A_1), \quad \text{if } B_{11,3}^1 = 0,$$

$$(A_{ij}) = \text{diag}(A_1, A_1, A_1, -A_1, \dots, -A_1), \quad \text{if } B_{11,3}^1 \neq 0.$$

Particularly, if  $B_{11,3}^1 \neq 0$ ,  $\omega_{3i} = 0, i \geq 4$  and  $A_1 = \frac{(B_{11,3}^1)^2}{2\mu^2}$ .



## 6 A canonical form of the normal connections

Before going into the detail, we observe that among the basic examples, the Veronese surfaces in either  $S^{2k}$  or  $\mathbb{C}P^k$  have a well-known property as being totally isotropic. In particular, the normal bundle of any of these examples has a decomposition into a series of 2-planes, and the complex normal bundle has a corresponding decomposition into isotropic complex lines. This beautiful structure is also shared by the Wintgen ideal submanifolds constructed from them by generating the cones.

In this section we will show that in the most important cases, a Möbius homogeneous Wintgen ideal submanifold must have a similar decomposition of the normal bundle. The consequence is that they normal connections can take a canonical form with respect to a good normal frames as demonstrated by two propositions below.

**Proposition 6.1.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $B_{11,3}^1 = 0$  and  $\omega_{12} \neq 0$ , then there exist basis  $\{\xi_3, \xi_4, \dots, \xi_p\}$  in  $\text{Span}\{\xi_3, \xi_4, \dots, \xi_p\}$  such that the normal connection have the following form*

$$\begin{aligned} & \left\{ \begin{array}{l} \left\{ \begin{array}{l} \theta_{13} = a_0 \omega_2, \\ \theta_{23} = -a_0 \omega_1, \end{array} \right. \left\{ \begin{array}{l} \theta_{14} = a_0 \omega_1, \\ \theta_{24} = a_0 \omega_2, \end{array} \right. \left\{ \begin{array}{l} \theta_{1\alpha} = 0, \quad \alpha \geq 5, \\ \theta_{2\alpha} = 0, \quad \alpha \geq 5, \end{array} \right. \\ \theta_{12} - \omega_{12} = \theta_{34}, \quad a_0 \neq 0; \end{array} \right. \\ & \left\{ \begin{array}{l} \left\{ \begin{array}{l} \theta_{(2k+1)(2k+3)} = a_k \omega_2, \\ \theta_{(2k+2)(2k+3)} = -a_k \omega_1, \end{array} \right. \left\{ \begin{array}{l} \theta_{(2k+1)(2k+4)} = a_k \omega_1, \\ \theta_{(2k+2)(2k+4)} = a_k \omega_2, \end{array} \right. \left\{ \begin{array}{l} \theta_{(2k+1)\alpha} = 0, \quad \alpha \geq 2k+5, \\ \theta_{(2k+2)\alpha} = 0, \quad \alpha \geq 2k+5, \end{array} \right. \\ \theta_{(2k+1)(2k+2)} - \omega_{12} = \theta_{(2k+3)(2k+4)}, \quad a_k \neq 0; \end{array} \right. \\ & \left\{ \begin{array}{l} \theta_{(2k+3)\alpha} = 0, \quad \alpha \geq 2k+5, \\ \theta_{(2k+4)\alpha} = 0, \quad \alpha \geq 2k+5, \end{array} \right. \end{aligned}$$

*Proof.* Since  $B_{11,3}^1 = 0$ , from (5.53), we have

$$(6.54) \quad 2\omega_{12} + \theta_{12} = 0, \quad d\omega_1 = \omega_{12} \wedge \omega_2, \quad d\omega_2 = -\omega_{12} \wedge \omega_1.$$

We assume  $p \geq 3$ . We can rechoose an orthonormal basis in  $\text{Span}\{\xi_3, \dots, \xi_p\}$  such that

$$(6.55) \quad \left\{ \begin{array}{l} \theta_{13} = \frac{B_{11,2}^3}{\mu} \omega_1 + \frac{B_{22,1}^3}{\mu} \omega_2, \\ \theta_{23} = -\frac{B_{22,1}^3}{\mu} \omega_1 + \frac{B_{11,2}^3}{\mu} \omega_2, \end{array} \right. \left\{ \begin{array}{l} \theta_{14} = \frac{B_{11,2}^4}{\mu} \omega_1, \\ \theta_{24} = \frac{B_{11,2}^4}{\mu} \omega_2, \end{array} \right. \left\{ \begin{array}{l} \theta_{1\alpha} = 0, \quad \alpha \geq 5, \\ \theta_{2\alpha} = 0, \quad \alpha \geq 5, \end{array} \right.$$

Using  $d\theta_{13} - \sum_{\tau} \theta_{1\tau} \wedge \theta_{\tau 3} = -\frac{1}{2} \sum_{kl} R_{13kl}^\perp \omega_k \wedge \omega_l$  and (6.55), and noting  $R_{13kl}^\perp = 0$ , we obtain

$$(6.56) \quad \frac{B_{11,2}^3}{\mu} \Theta \wedge \omega_2 - \frac{B_{22,1}^3}{\mu} \Theta \wedge \omega_1 + \frac{B_{11,2}^4}{\mu} \theta_{34} \wedge \omega_1 = 0, \quad \Theta = \theta_{12} - \omega_{12}.$$

Similarly, we have

$$(6.57) \quad \begin{aligned} & \frac{B_{22,1}^3}{\mu} \Theta \wedge \omega_2 + \frac{B_{11,2}^3}{\mu} \Theta \wedge \omega_1 - \frac{B_{11,2}^4}{\mu} \theta_{34} \wedge \omega_2 = 0, \\ & \frac{B_{11,2}^4}{\mu} \Theta \wedge \omega_1 - \left( -\frac{B_{22,1}^3}{\mu} \omega_1 + \frac{B_{11,2}^3}{\mu} \omega_2 \right) \wedge \theta_{34} = 0, \\ & \frac{B_{11,2}^4}{\mu} \Theta \wedge \omega_2 + \left( \frac{B_{11,2}^3}{\mu} \omega_1 + \frac{B_{22,1}^3}{\mu} \omega_2 \right) \wedge \theta_{34} = 0. \end{aligned}$$

From (6.56) and (6.57), we eliminate the terms  $\omega_1 \wedge \theta_{34}$  and  $\omega_2 \wedge \theta_{34}$ , and we get

$$(6.58) \quad \begin{aligned} & 2 \frac{B_{22,1}^3 B_{11,2}^3}{\mu^2} \Theta \wedge \omega_2 + \left[ \left( \frac{B_{11,2}^3}{\mu} \right)^2 - \left( \frac{B_{22,1}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^4}{\mu} \right)^2 \right] \Theta \wedge \omega_1 = 0, \\ & \left[ \left( \frac{B_{11,2}^3}{\mu} \right)^2 - \left( \frac{B_{22,1}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^4}{\mu} \right)^2 \right] \Theta \wedge \omega_2 - 2 \frac{B_{22,1}^3 B_{11,2}^3}{\mu^2} \Theta \wedge \omega_1 = 0. \end{aligned}$$

Let  $D = [2 \frac{B_{22,1}^3 B_{11,2}^3}{\mu^2}]^2 + [(\frac{B_{11,2}^3}{\mu})^2 - (\frac{B_{22,1}^3}{\mu})^2 + (\frac{B_{11,2}^4}{\mu})^2]^2$ .

If  $D \neq 0$ , then, from (6.58), we have

$$\Theta = \theta_{12} - \omega_{12} = 0.$$

Combining (6.54), we have  $\omega_{12} = 0$ , which is in contradiction with the assumption  $\omega_{12} \neq 0$ . Thus  $D = 0$ .

If  $B_{22,1}^3 = 0$ , noting  $D = 0$ , we get  $B_{11,2}^3 = B_{11,2}^4 = 0$ .

$$\begin{cases} \theta_{1\alpha} = 0, & \alpha \geq 3, \\ \theta_{2\alpha} = 0, & \alpha \geq 3, \end{cases}$$

Thus we finish the proof.

If  $B_{22,1}^3 \neq 0$ , we get  $B_{11,2}^3 = 0$  and  $(B_{22,1}^3)^2 = (B_{11,2}^4)^2$ . Let  $B_{22,1}^3 = B_{11,2}^4 \triangleq \mu a_0$ , otherwise, take  $\tilde{\xi}_4 = -\xi_4$ . From (6.55), we have

$$\begin{cases} \theta_{13} = a_0 \omega_2, & \begin{cases} \theta_{14} = a_0 \omega_1, \\ \theta_{23} = -a_0 \omega_1, \end{cases} & \begin{cases} \theta_{1\alpha} = 0, & \alpha \geq 5, \\ \theta_{2\alpha} = 0, & \alpha \geq 5, \end{cases} \\ \theta_{12} - \omega_{12} = \theta_{34}. \end{cases}$$

Combining the above formula with  $d\theta_{1\alpha} = \sum_{\tau} \theta_{1\tau} \wedge \theta_{\tau\alpha}$ , we have

$$\omega_2 \wedge \theta_{3\alpha} + \omega_1 \wedge \omega_{4\alpha} = 0, \quad \alpha \geq 5, \quad -\omega_1 \wedge \theta_{3\alpha} + \omega_2 \wedge \omega_{4\alpha} = 0, \quad \alpha \geq 5.$$

Thus we can assume that

$$(6.59) \quad \begin{cases} \theta_{3\alpha} = a_1^\alpha \omega_1 + a_2^\alpha \omega_2, \\ \theta_{4\alpha} = -a_2^\alpha \omega_1 + a_1^\alpha \omega_2, \end{cases} \quad \alpha \geq 5.$$

We can make a new choice of orthonormal frames in  $Span\{\xi_5, \dots, \xi_p\}$  such that

$$(6.60) \quad \begin{cases} \theta_{35} = a\omega_1 + b\omega_2, & \begin{cases} \theta_{36} = c\omega_1, \\ \theta_{45} = -b\omega_1 + a\omega_2, \end{cases} & \begin{cases} \theta_{3\alpha} = 0, & \alpha \geq 7, \\ \theta_{4\alpha} = 0, & \alpha \geq 7. \end{cases} \end{cases}$$

Using  $d\theta_{\alpha\beta} = \sum_{\gamma} \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta}$  for  $\alpha = 3, 4, \beta = 5, 6$  and (6.60), we can obtain that

$$(6.61) \quad \begin{aligned} & a\Theta \wedge \omega_2 - b\Theta \wedge \omega_1 - c\omega_1 \wedge \theta_{56} = 0, \\ & b\Theta \wedge \omega_2 + a\Theta \wedge \omega_1 + c\omega_2 \wedge \theta_{56} = 0, \\ & c\Theta \wedge \omega_2 + [a\omega_1 + b\omega_2] \wedge \theta_{56} = 0, \\ & c\Theta \wedge \omega_1 - [-b\omega_1 + a\omega_2] \wedge \theta_{56} = 0, \end{aligned}$$

where  $\Theta = \theta_{34} - \omega_{12}$ .

From (6.61), we have

$$(6.62) \quad \begin{cases} 2ab\Theta \wedge \omega_1 + (b^2 - a^2 - c^2)\Theta \wedge \omega_2 = 0, \\ (a^2 - b^2 + c^2)\Theta \wedge \omega_1 + 2ab\Theta \wedge \omega_2 = 0. \end{cases}$$

If  $D = 4a^2b^2 + [a^2 - b^2 + c^2]^2 \neq 0$ , then (6.62) implies that

$$\Theta = \theta_{34} - \omega_{12} = 0.$$

From (6.54)  $2\omega_{12} + \theta_{12} = 0$  and  $\theta_{34} = \theta_{12} - \omega_{12}$ , we obtain that  $\omega_{12} = 0$ , which is in contradiction with the assumption  $\omega_{12} \neq 0$ . Therefore

$$D = 4a^2b^2 + [a^2 - b^2 + c^2]^2 = 0.$$

If  $b = 0$ , combining with  $D = 4a^2b^2 + [a^2 - b^2 + c^2]^2 = 0$ , we have  $a = c = 0$ . Thus we finish the proof.

If  $b \neq 0$ , then combining with  $D = 4a^2b^2 + [a^2 - b^2 + c^2]^2 = 0$ , we have  $a = 0$  and  $b = \pm c$ . Let  $b = c \triangleq a_1$ , otherwise  $\tilde{\xi}_6 = -\xi_6$ . From (6.60), we have

$$(6.63) \quad \begin{cases} \theta_{35} = a_1\omega_2, \\ \theta_{45} = -a_1\omega_1, \end{cases} \begin{cases} \theta_{36} = a_1\omega_1, \\ \theta_{46} = a_1\omega_2, \end{cases} \begin{cases} \theta_{3\alpha} = 0, & \alpha \geq 7, \\ \theta_{4\alpha} = 0, & \alpha \geq 7, \end{cases} \quad \theta_{56} = \theta_{34} - \omega_{12}.$$

Repeating the process (6.59–6.63), we finish the proof.  $\square$

**Proposition 6.2.** *Let  $f : M^3 \longrightarrow \mathbb{R}^{3+p}$  be a Möbius homogeneous Wintgen ideal submanifold. If  $B_{11,3}^1 \neq 0$ , then there exist basis  $\{\xi_3, \xi_4, \dots, \xi_p\}$  in  $\text{Span}\{\xi_3, \xi_4, \dots, \xi_p\}$  such that the normal connection have the following form*

$$\begin{cases} \begin{cases} \theta_{13} = a_0\omega_2, \\ \theta_{23} = -a_0\omega_1, \end{cases} \begin{cases} \theta_{14} = a_0\omega_1, \\ \theta_{24} = a_0\omega_2, \end{cases} \begin{cases} \theta_{1\alpha} = 0, & \alpha \geq 5, \\ \theta_{2\alpha} = 0, & \alpha \geq 5, \end{cases} \\ \theta_{12} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3 = \theta_{34}, \quad a_0 \neq 0; \\ \begin{cases} \begin{cases} \theta_{(2k+1)(2k+3)} = a_k\omega_2, \\ \theta_{(2k+2)(2k+3)} = -a_k\omega_1, \end{cases} \begin{cases} \theta_{(2k+1)(2k+4)} = a_k\omega_1, \\ \theta_{(2k+2)(2k+4)} = a_k\omega_2, \end{cases} \begin{cases} \theta_{(2k+1)\alpha} = 0, & \alpha \geq 2k+5, \\ \theta_{(2k+2)\alpha} = 0, & \alpha \geq 2k+5, \end{cases} \\ \theta_{(2k+1)(2k+2)} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3 = \theta_{(2k+3)(2k+4)}, \quad a_k \neq 0, \\ \begin{cases} \theta_{(2k+3)\alpha} = 0, & \alpha \geq 2k+5, \\ \theta_{(2k+4)\alpha} = 0, & \alpha \geq 2k+5. \end{cases} \end{cases}$$

*Proof.* We use induction for  $k$  to prove the Proposition 6.2. Choose a new orthonormal basis in  $\text{Span}\{\xi_3, \dots, \xi_p\}$  such that

$$(6.64) \quad \begin{cases} \theta_{13} = \frac{B_{11,2}^3}{\mu}\omega_1 + \frac{B_{22,1}^3}{\mu}\omega_2, \\ \theta_{23} = -\frac{B_{22,1}^3}{\mu}\omega_1 + \frac{B_{11,2}^3}{\mu}\omega_2, \end{cases} \begin{cases} \theta_{14} = \frac{B_{11,2}^4}{\mu}\omega_1, \\ \theta_{24} = \frac{B_{11,2}^4}{\mu}\omega_2, \end{cases} \begin{cases} \theta_{1\alpha} = 0, & \alpha \geq 5, \\ \theta_{2\alpha} = 0, & \alpha \geq 5, \end{cases}$$

Using  $d\theta_{\alpha\beta} - \sum_{\tau} \theta_{\alpha\tau} \wedge \theta_{\tau\beta} = -\frac{1}{2} \sum_{kl} R_{\alpha\beta kl}^\perp \omega_k \wedge \omega_l$  and (6.64), and noting  $R_{\alpha\beta kl}^\perp = 0$  for  $\alpha = 1, 2, \beta = 3, 4$ , we obtain

$$(6.65) \quad \begin{aligned} & \frac{B_{11,2}^3}{\mu}\Theta \wedge \omega_2 - \frac{B_{22,1}^3}{\mu}\Theta \wedge \omega_1 + \frac{B_{11,2}^4}{\mu}\theta_{34} \wedge \omega_1 = 0, \\ & \frac{B_{22,1}^3}{\mu}\Theta \wedge \omega_2 + \frac{B_{11,2}^3}{\mu}\Theta \wedge \omega_1 - \frac{B_{11,2}^4}{\mu}\theta_{34} \wedge \omega_2 = 0, \\ & \frac{B_{11,2}^4}{\mu}\Theta \wedge \omega_1 - \left( -\frac{B_{22,1}^3}{\mu}\omega_1 + \frac{B_{11,2}^3}{\mu}\omega_2 \right) \wedge \theta_{34} = 0, \\ & \frac{B_{11,2}^4}{\mu}\Theta \wedge \omega_2 + \left( \frac{B_{11,2}^3}{\mu}\omega_1 + \frac{B_{22,1}^3}{\mu}\omega_2 \right) \wedge \theta_{34} = 0, \end{aligned}$$

where  $\Theta = \theta_{12} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3$ ,

From (6.65), we eliminate the terms  $\omega_1 \wedge \theta_{34}$  and  $\omega_2 \wedge \theta_{34}$ , and we get

$$(6.66) \quad \begin{aligned} & 2\frac{B_{22,1}^3 B_{11,2}^3}{\mu^2} \Theta \wedge \omega_2 + \left[ \left( \frac{B_{11,2}^3}{\mu} \right)^2 - \left( \frac{B_{22,1}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^4}{\mu} \right)^2 \right] \Theta \wedge \omega_1 = 0, \\ & \left[ \left( \frac{B_{11,2}^3}{\mu} \right)^2 - \left( \frac{B_{22,1}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^4}{\mu} \right)^2 \right] \Theta \wedge \omega_2 - 2\frac{B_{22,1}^3 B_{11,2}^3}{\mu^2} \Theta \wedge \omega_1 = 0. \end{aligned}$$

**Claim1:**  $D = [2\frac{B_{22,1}^3 B_{11,2}^3}{\mu^2}]^2 + [(\frac{B_{11,2}^3}{\mu})^2 - (\frac{B_{22,1}^3}{\mu})^2 + (\frac{B_{11,2}^4}{\mu})^2] = 0$ .

Proof of Claim1: Since  $|B|^2 = \frac{m-1}{m}$ , using covariant derivative of  $B_{ij,k}^\alpha$  and  $\Phi = 0$ , we have

$$(6.67) \quad \begin{aligned} 0 = \frac{1}{2} \Delta |B|^2 &= \sum_{\alpha, i, j, k} |B_{ij,k}^\alpha|^2 + \sum_{\alpha, i, j, k, l} B_{ij}^\alpha B_{kl}^\alpha R_{kijl} \\ &+ \sum_{\alpha, i, j, k, l} B_{ij}^\alpha B_{ik}^\alpha R_{jlk} - \sum_{\alpha, \beta, i, j, k} B_{ij}^\alpha B_{ki}^\beta R_{\alpha\beta jk}^\perp. \end{aligned}$$

If  $D \neq 0$ , from (6.79) we have  $\Theta = 0$ ,  $\theta_{12} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3 = 0$ . Combining  $2\omega_{12} + \theta_{12} + \frac{B_{11,3}^1}{\mu}\omega_3 = 0$ , we have

$$(6.68) \quad \omega_{12} = -\frac{2B_{11,3}^1}{3\mu}\omega_3, \quad \theta_{12} = \frac{B_{11,3}^1}{3\mu}\omega_3.$$

Using  $d\omega_3 = -2\frac{B_{11,3}^1}{\mu}\omega_1 \wedge \omega_2$ ,  $d\omega_{12} - \sum_k \omega_{1k} \wedge \omega_{k2} = -\frac{1}{2} \sum_{kl} R_{12kl} \omega_k \wedge \omega_l$  and  $d\theta_{12} - \sum_k \theta_{1k} \wedge \theta_{k2} = -\frac{1}{2} \sum_{kl} R_{12kl}^\perp \omega_k \wedge \omega_l$ , we obtain

$$(6.69) \quad R_{1212} = -\frac{7}{3} \left( \frac{B_{11,3}^1}{\mu} \right)^2, \quad R_{1212}^\perp = \frac{2}{3} \left( \frac{B_{11,3}^1}{\mu} \right)^2 - \left( \frac{B_{22,1}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^4}{\mu} \right)^2.$$

Combining (2.6), we have

$$(6.70) \quad \mu^2 = \frac{5}{3} \left( \frac{B_{11,3}^1}{\mu} \right)^2, \quad 4 \left( \frac{B_{11,3}^1}{\mu} \right)^2 = \left( \frac{B_{22,1}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^3}{\mu} \right)^2 + \left( \frac{B_{11,2}^4}{\mu} \right)^2.$$

Combining (6.67), (6.69) and (6.70), we get  $\mu = 0$ , which is a contradiction. so  $D = 0$ .

If  $B_{22,1}^3 = 0$ . Since  $D = 0$ , then  $B_{11,2}^3 = B_{11,2}^4 = 0$ , that is

$$\theta_{1\alpha} = 0, \quad \alpha \geq 3, \quad \theta_{2\alpha} = 0, \quad \alpha \geq 3.$$

This implies that the codimension of  $f$  can reduce to 2 and  $a_k = 0$  for  $k = 0$ .

If  $B_{22,1}^3 \neq 0$ . Since  $D = 0$ , then  $B_{11,2}^3 = 0$  and  $(B_{22,1}^3)^2 = (B_{11,2}^4)^2$ . We can assume that  $B_{22,1}^3 = B_{11,2}^4$ , otherwise, take  $\tilde{\xi}_4 = -\xi_4$ . Let  $a_0 = \frac{B_{11,2}^4}{\mu}$ . Thus

$$\begin{cases} \theta_{13} = a_0 \omega_2, \\ \theta_{23} = -a_0 \omega_1, \end{cases} \quad \begin{cases} \theta_{14} = a_0 \omega_1, \\ \theta_{24} = a_0 \omega_2, \end{cases} \quad \begin{cases} \theta_{1\alpha} = 0, & \alpha \geq 5, \\ \theta_{2\alpha} = 0, & \alpha \geq 5, \end{cases}$$

From (6.65), we get

$$\theta_{34} = \Theta = \theta_{12} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3.$$

Next we assume

$$(6.71) \quad \begin{cases} \theta_{(2k-1)(2k+1)} = a_{k-1}\omega_2, & \begin{cases} \theta_{(2k-1)(2k+2)} = a_{k-1}\omega_1, \\ \theta_{(2k)(2k+1)} = -a_{k-1}\omega_1, \end{cases} \\ \theta_{(2k)(2k+1)} = -a_{k-1}\omega_1, & \theta_{(2k)(2k+2)} = a_{k-1}\omega_2, \end{cases}$$

$$(6.72) \quad \begin{cases} \theta_{(2k-1)\alpha} = 0, & \alpha \geq 2k+3, \\ \theta_{(2k)\alpha} = 0, & \alpha \geq 2k+3, \end{cases}$$

$$(6.73) \quad \theta_{(2k-1)(2k)} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3 = \theta_{(2k+1)(2k+2)}.$$

Using  $d\theta_{(2k-1)\alpha} - \sum_{\tau} \theta_{(2k-1)\tau} \wedge \theta_{\tau\alpha} = -\frac{1}{2} \sum_{sl} R_{(2k-1)\alpha sl}^{\perp} \omega_s \wedge \omega_l$  and (6.72), and noting  $R_{(2k-1)\alpha sl}^{\perp} = 0$ , we obtain

$$(6.74) \quad \begin{cases} \omega_2 \wedge \theta_{(2k+1)\alpha} + \omega_1 \wedge \theta_{(2k+2)\alpha} = 0, & \alpha \geq 2k+3, \\ -\omega_1 \wedge \theta_{(2k+1)\alpha} + \omega_2 \wedge \theta_{(2k+2)\alpha} = 0, & \alpha \geq 2k+3. \end{cases}$$

From (6.74), we can assume that

$$\begin{cases} \theta_{(2k+1)\alpha} = a_1^{\alpha}\omega_1 + a_2^{\alpha}\omega_2, & \alpha \geq 2k+3, \\ \theta_{(2k+2)\alpha} = -a_2^{\alpha}\omega_1 + a_1^{\alpha}\omega_2, & \alpha \geq 2k+3. \end{cases}$$

Furthermore, we can choose a basis  $\{\xi_{2k+3}, \dots, \xi_p\}$  in  $Span\{\xi_{2k+3}, \dots, \xi_p\}$  such that

$$(6.75) \quad \begin{cases} \theta_{(2k+1)(2k+3)} = a\omega_1 + b\omega_2, \\ \theta_{(2k+2)(2k+3)} = -b\omega_1 + a\omega_2, \end{cases}$$

$$(6.76) \quad \begin{cases} \theta_{(2k+1)(2k+4)} = c\omega_1, & \begin{cases} \theta_{(2k+1)\alpha} = 0, & \alpha \geq 2k+5, \\ \theta_{(2k+2)(2k+4)} = c\omega_2, & \theta_{(2k+2)\alpha} = 0, & \alpha \geq 2k+5. \end{cases} \end{cases}$$

Using  $d\theta_{(2k+1)(2k+3)} - \sum_{\tau} \theta_{(2k+1)\tau} \wedge \theta_{\tau(2k+3)} = -\frac{1}{2} \sum_{sl} R_{(2k+1)(2k+3)sl}^{\perp} \omega_s \wedge \omega_l$  and (6.75), and noting  $R_{(2k+1)(2k+3)sl}^{\perp} = 0$ , we obtain

$$(6.77) \quad a\Theta \wedge \omega_2 - b\Theta \wedge \omega_1 + c\theta_{(2k+3)(2k+4)} \wedge \omega_1 = 0, \quad \Theta = \theta_{(2k+1)(2k+2)} - \omega_{12} - \frac{B_{11,3}^1}{\mu}\omega_3.$$

Similarly, we have

$$(6.78) \quad \begin{aligned} b\Theta \wedge \omega_2 + a\Theta \wedge \omega_1 - c\theta_{(2k+3)(2k+4)} \wedge \omega_2 &= 0, \\ c\Theta \wedge \omega_1 - (-b\omega_1 + a\omega_2) \wedge \theta_{(2k+3)(2k+4)} &= 0, \\ c\Theta \wedge \omega_2 + (a\omega_1 + b\omega_2) \wedge \theta_{(2k+3)(2k+4)} &= 0. \end{aligned}$$

From (6.77) and (6.78), we eliminate the terms  $\omega_1 \wedge \theta_{(2k+3)(2k+4)}$  and  $\omega_2 \wedge \theta_{(2k+3)(2k+4)}$ , and we get

$$(6.79) \quad \begin{aligned} 2ab\Theta \wedge \omega_2 + [a^2 - b^2 + c^2]\Theta \wedge \omega_1 &= 0, \\ [a^2 - b^2 + c^2]\Theta \wedge \omega_2 - 2ab\Theta \wedge \omega_1 &= 0. \end{aligned}$$

Like Claim 1, we can prove that

$$D = [2ab]^2 + [a^2 - b^2 + c^2]^2 = 0.$$

If  $b = 0$ . Since  $D = 0$ , then  $a = c = 0$ , which implies that the codimension of  $f$  can reduce to  $2k + 2$  and  $a_k = 0$ .

If  $b \neq 0$ . Since  $D = 0$ , then  $a = 0$  and  $b^2 = c^2$ . We can assume  $b = c = a_k$ , thus

$$\begin{cases} \theta_{(2k+1)(2k+3)} = a_k \omega_2, & \begin{cases} \theta_{(2k+1)(2k+4)} = a_k \omega_1, \\ \theta_{(2k+2)(2k+3)} = -a_k \omega_1, \end{cases} & \begin{cases} \theta_{(2k+1)\alpha} = 0, \quad \alpha \geq 2k+5, \\ \theta_{(2k+2)\alpha} = 0, \quad \alpha \geq 2k+5, \end{cases} \end{cases}$$

Since  $a = 0$  and  $b = c$ , From (6.77) we have

$$\theta_{(2k+3)(2k+4)} = \Theta = \theta_{(2k+1)(2k+2)} - \omega_{12} - \frac{B_{11,3}^1}{\mu} \omega_3.$$

Thus we finish the proof to Proposition 6.2.  $\square$

## 7 The reduction to 2 and 3 dimensional case

To obtain the classification of Möbius homogeneous Wintgen ideal submanifolds, a crucial observation is that they can be reduced to 2 or 3 dimensional case via the construction of cones in Section 4. The reader may compare this to our general results for Wintgen ideal submanifolds with a low-dimensional integrable distribution [17], and the reduction theorem for hypersurfaces in Möbius geometry [16].

**Lemma 7.1.** *Let  $f : M^m \longrightarrow \mathbb{R}^{m+p}$  ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. Then  $A_1 > 0$ .*

*Proof.* If  $B_{11,3}^1 \neq 0$ , from Proposition 5.6,  $A_1 = \frac{(B_{11,3}^1)^2}{2\mu^2} > 0$ . Thus we need only to consider the case when  $B_{11,3}^1 = 0$ .

If  $B_{11,3}^1 = 0, \omega_{12} = 0$ , then  $R_{1212} = 0$ , i.e.,  $-2\mu^2 + 2A_1 = 0$ . Thus  $A_1 = \mu^2 > 0$ .

Consider the case  $B_{11,3}^1 = 0, \omega_{12} \neq 0$ . From Proposition 6.1, we have

$$(7.80) \quad \theta_{12} - (k+1)\omega_{12} = \theta_{(2k+3)(2k+4)}.$$

From (5.53), we have

$$(7.81) \quad 2\omega_{12} + \theta_{12} = 0, \quad \omega_{1i} = 0, i \geq 3, \quad \omega_{2i} = 0, i \geq 3.$$

Thus, we have

$$(7.82) \quad \theta_{(2k+3)(2k+4)} = -(k+3)\omega_{12}.$$

Using Proposition (6.1) and (7.82), we have

$$d\theta_{(2k+3)(2k+4)} = \sum_{\tau} \theta_{(2k+3)\tau} \wedge \theta_{\tau(2k+4)} = 2a_k^2 \omega_1 \wedge \omega_2 = (k+3)R_{1212} \omega_1 \wedge \omega_2,$$

which implies

$$(7.83) \quad 2a_k^2 = (k+2)R_{1212} = (k+3)(-2\mu^2 + 2A_1).$$

Thus  $A_1 > 0$ .  $\square$

**Proposition 7.2.** *Let  $f : M^m \longrightarrow \mathbb{R}^{m+p}$  ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $B_{11,3}^1 = 0$ , then locally  $f$  is Möbius equivalent to a cone over a surface  $u : M^2 \rightarrow \mathbb{S}^{2+p}$ .*

*Proof.* From (5.53), we have

$$(7.84) \quad 2\omega_{12} + \theta_{12} = 0, \quad \omega_{1i} = 0, i \geq 3, \quad \omega_{2i} = 0, i \geq 3.$$

Since  $d\omega_a \equiv 0, \text{mod}\{\omega_3, \dots, \omega_m\}$ ,  $a \geq 3$ ,  $\mathbb{D} = \text{span}\{E_1, E_2\}$  is integrable. Using (7.84) and Proposition 5.6, we have

$$(7.85) \quad \begin{aligned} d\xi_1 &= -\mu\omega_1Y_2 - \mu\omega_2Y_1 + \sum_{\alpha=1} \theta_{1\alpha}\xi_\alpha, \\ d\xi_2 &= -\mu\omega_1Y_1 + \mu\omega_2Y_2 + \sum_{\alpha=1} \theta_{2\alpha}\xi_\alpha, \\ d\xi_\alpha &= -\theta_{1\alpha}\xi_1 - \theta_{2\alpha}\xi_2 + \sum_{\beta} \theta_{\alpha\beta}\xi_\beta. \end{aligned}$$

$$(7.86) \quad \begin{aligned} dY_1 &= \omega_{12}Y_2 - \omega_1(A_1Y + N) + \mu\omega_2\xi_1 + \mu\omega_1\xi_2, \\ dY_2 &= -\omega_{12}Y_1 - \omega_2(A_1Y + N) + \mu\omega_1\xi_1 - \mu\omega_2\xi_2, \\ d(A_1Y + N) &= 2A_1[\omega_1Y_1 + \omega_2Y_2] \end{aligned}$$

From (7.85) and (7.86), we know that the subspace

$$V = \text{span}\{(A_1Y + N), Y_1, Y_2, \xi_1, \xi_2, \dots, \xi_p\}$$

is parallel along  $M^m$ . The orthogonal complement  $V^\perp$  also is parallel along  $M^m$ . In fact,

$$V^\perp = \text{span}\{(A_1Y - N), Y_3, \dots, Y_m\}.$$

Using (7.84), we can obtain

$$(7.87) \quad d(A_1Y - N) = 2A_1 \sum_{a \geq 3} \omega_a Y_a, \quad dY_a = \omega_a(A_1Y - N) + \sum_{b \geq 3} \omega_{ab} Y_b, \quad a \geq 3.$$

Since  $d\omega_1 = \omega_{12} \wedge \omega_1$ ,  $d\omega_2 = -\omega_{12} \wedge \omega_1$ , the distribution  $\mathbb{D}^\perp = \text{span}\{E_3, \dots, E_m\}$  also is integrable. From (7.85) and (7.86), we know that the mean curvature spheres  $\xi_1, \xi_2$  induce 2-dimensional submanifolds in the de sitter space  $\mathbb{S}_1^{m+p+1}$

$$\xi_1, \xi_2 : M^2 = M^m/F \longrightarrow \mathbb{S}_1^{m+p+1},$$

where fibers  $F$  are integral submanifolds of distribution  $\mathbb{D}^\perp$ . In other words,  $\xi_1, \xi_2$  form 2-parameter family of  $(m+p-1)$ -spheres enveloped by  $f : M^m \longrightarrow \mathbb{R}^{m+p}$ .

Since  $\langle (A_1Y + N), (A_1Y + N) \rangle = 2A_1 > 0$ ,  $V$  is a fixed space-like subspace,  $V^\perp$  is a fixed Lorentz subspace in  $\mathbb{R}_1^{m+p+2}$ . We can assume that  $V = \mathbb{R}^{3+p}$ ,  $V^\perp = \mathbb{R}_1^{m-1}$ . From (7.85) and (7.86), we know

$$u = \frac{1}{\sqrt{A_1}}(A_1Y + N) : M^2 \rightarrow \mathbb{S}^{2+p}.$$

On the other hand, the equation (7.87) implies that

$$\phi = \frac{1}{\sqrt{A_1}}(A_1Y - N) : \mathbb{H}^{m-2} \rightarrow \mathbb{R}_1^{m-1}$$

is the standard embedding of the hyperbolic space  $\mathbb{H}^{m-2}$  in  $\mathbb{R}_1^{m-1}$ . Then

$$Y = 2\sqrt{A_1}(u, \phi) : M^2 \times \mathbb{H}^{m-2} \rightarrow \mathbb{S}^{2+p} \times \mathbb{H}^{m-2} \subset \mathbb{R}_1^{m+p+2},$$

where  $\phi : \mathbb{H}^{m-2} \rightarrow \mathbb{H}^{m-2}$  is the identity map. From Proposition 4.3, we know that  $f$  is a cone over  $u : M^2 \rightarrow \mathbb{S}^{2+p}$ . We complete the proof of Proposition 7.2.  $\square$

**Proposition 7.3.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  ( $m \geq 4$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $B_{11,3}^1 \neq 0$ , then locally  $f$  is Möbius equivalent to a cone over a three dimensional Möbius homogeneous Wintgen ideal submanifold in  $\mathbb{S}^{3+p}$ .*

*Proof.* From (5.53) and Proposition 5.6, we have

$$(7.88) \quad \begin{aligned} 2\omega_{12} + \theta_{12} &= \frac{-B_{11,3}^1}{\mu}\omega_3, \quad \omega_{13} = \frac{-B_{11,3}^1}{\mu}\omega_2, \quad \omega_{23} = \frac{B_{11,3}^1}{\mu}\omega_1, \\ \omega_{1i} &= 0, i \geq 4, \quad \omega_{2i} = 0, i \geq 4, \quad \omega_{3i} = 0, i \geq 4. \end{aligned}$$

Since  $d\omega_a \equiv 0, \text{mod}\{\omega_4, \dots, \omega_m\}$ ,  $a \geq 4$ ,  $\mathbb{D} = \text{span}\{E_1, E_2, E_3\}$  is integrable, Using (7.88) and Proposition 5.6, we have

$$(7.89) \quad \begin{aligned} d\xi_1 &= -\mu\omega_1Y_2 - \mu\omega_2Y_1 + \sum_{\alpha=1} \theta_{1\alpha}\xi_\alpha, \\ d\xi_2 &= -\mu\omega_1Y_1 + \mu\omega_2Y_2 + \sum_{\alpha=1} \theta_{2\alpha}\xi_\alpha, \\ d\xi_\alpha &= -\theta_{1\alpha}\xi_1 - \theta_{2\alpha}\xi_2 + \sum_{\beta} \theta_{\alpha\beta}\xi_\beta. \end{aligned}$$

$$(7.90) \quad \begin{aligned} dY_1 &= -\omega_1(A_1Y + N) + \omega_{12}Y_2 + \omega_{13}Y_3 + \mu\omega_2\xi_1 + \mu\omega_1\xi_2, \\ dY_2 &= -\omega_2(A_1Y + N) - \omega_{12}Y_1 + \omega_{23}Y_3 + \mu\omega_1\xi_1 - \mu\omega_2\xi_2, \\ dY_3 &= -\omega_3(A_1Y + N) - \omega_{13}Y_1 - \omega_{23}Y_2, \\ d(A_1Y + N) &= 2A_1[\omega_1Y_1 + \omega_2Y_2 + \omega_3Y_3]. \end{aligned}$$

From (7.89) and (7.90), we know that the subspace

$$V = \text{span}\{(A_1Y + N), Y_1, Y_2, Y_3, \xi_1, \xi_2, \dots, \xi_p\}$$

is parallel along  $M^m$ . The orthogonal complement  $V^\perp$  also is parallel along  $M^m$ . In fact,

$$V^\perp = \text{span}\{(A_1Y - N), Y_4, \dots, Y_m\}.$$

Using (7.84), we can obtain

$$(7.91) \quad d(A_1Y - N) = 2A_1 \sum_{a \geq 4} \omega_a Y_a, \quad dY_a = \omega_a(A_1Y - N) + \sum_{b \geq 4} \omega_{ab} Y_b, \quad a \geq 4.$$

Since  $\langle (A_1Y + N), (A_1Y + N) \rangle = 2A_1 > 0$ ,  $V$  is a fixed space-like subspace. Like as the proof of Proposition 7.2, we can prove that  $f$  is Möbius equivalent to a cone over a three dimensional Wintgen ideal submanifold  $u : M^3 \rightarrow \mathbb{S}^{3+p}$ . Since  $f$  is Möbius homogeneous, clearly  $u : M^3 \rightarrow \mathbb{S}^{3+p}$  is also Möbius homogeneous.  $\square$

## 8 Proof of the Main theorem

**Proposition 8.1.** *Let  $f : M^m \rightarrow \mathbb{R}^{m+p}$  ( $m \geq 3$ ) be a Möbius homogeneous Wintgen ideal submanifold. If  $B_{11,3}^1 = 0$ , then locally  $f$  is Möbius equivalent to*

- (i) *a cone over a Veronese surface in  $\mathbb{S}^{2k}$ ,*
- (ii) *a cone over a flat minimal surface in  $\mathbb{S}^{2k+1}$ .*



*Proof.* From Proposition 7.2, we know that  $f$  is a cone over  $u : M^2 \rightarrow \mathbb{S}^{2+p}$ . Since the Möbius form of  $f$  vanishes, from Proposition 4.4, we know that the Möbius form of the surface  $u$  vanishes. The surfaces with vanishing Möbius form is classified in [14]. We complete the proof to Proposition 8.1.  $\square$

If  $B_{11,3}^1 \neq 0$ , by Proposition 7.3 we need only to consider three dimensional Möbius homogeneous Wintgen ideal submanifolds in  $S^{3+p}$ .

**Proposition 8.2.** *Let  $x : M^3 \rightarrow \mathbb{S}^{3+p}$  be a Möbius homogeneous Wintgen ideal submanifold. If  $B_{11,3}^1 \neq 0$ , then locally  $x$  is Möbius equivalent to the Möbius homogeneous Wintgen ideal submanifold given by Example 1.4.*

*Proof.* Let  $\sigma : \mathbb{S}^{3+p} \rightarrow \mathbb{R}^{3+p}$  the stereographic projection. From [18], we know that the submanifolds  $x : M^3 \rightarrow \mathbb{S}^{3+p}$  and  $f = \sigma \circ x : M^3 \rightarrow \mathbb{R}^{3+p}$  have the same Möbius invariants, especially, the normal connection. From Proposition 6.2, we can assume  $f : M^3 \rightarrow \mathbb{S}^{2n+3}$  and there exists basis  $\{\xi_1, \xi_2, \dots, \xi_{2n}\}$  such that the normal connection under this frame has the following forms

$$\begin{cases} \theta_{(2k+1)(2k+3)} = a_k \omega_2, & \begin{cases} \theta_{(2k+1)(2k+4)} = a_k \omega_1, \\ \theta_{(2k+2)(2k+3)} = -a_k \omega_1, \end{cases} & \begin{cases} \theta_{(2k+1)\alpha} = 0, & 2k+5 \leq \alpha \leq 2n, \\ \theta_{(2k+2)\alpha} = 0, & 2k+5 \leq \alpha \leq 2n, \end{cases} \end{cases}$$

$$\theta_{34} = \theta_{12} - \omega_{12} - \frac{B_{11,3}^1}{\mu} \omega_3, \quad \theta_{(2k+3)(2k+4)} = \theta_{(2k+1)(2k+2)} - \omega_{12} - \frac{B_{11,3}^1}{\mu} \omega_3,$$

where  $k = 0, 1, 2, \dots, n-2$ .

From the preceding discussion, we know that when  $B_{11,3}^1 \neq 0$

$$\omega_{3i} = 0, i \geq 4; \quad (A_{ij}) = \text{diag}(A_1, A_1, A_1, -A_1, \dots, -A_1);$$

where  $A_1 = \frac{(B_{11,3}^1)^2}{2\mu^2}$ . Without lost of generality, we assume  $\sqrt{2A_1} = \frac{B_{11,3}^1}{\mu} \doteq L$ . In the following, we define

$$\eta = \frac{A_1 + N}{\sqrt{2A_1}}.$$

In fact we have  $\eta : M^3 \rightarrow \mathbb{S}^{2n+3}$ , this follows from the following structure equations (8.92).

Combining (7.89) and (7.90), with respect to the frame  $\{\eta, Y_3, Y_1, Y_2, \xi_1, \xi_2, \dots, \xi_{2n-1}, \xi_{2n}\}$  we can write out the structure equations as follows:

$$(8.92) \quad d \begin{pmatrix} \eta \\ Y_3 \\ Y_1 \\ Y_2 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{2n-1} \\ \xi_{2n} \end{pmatrix} = \Theta \begin{pmatrix} \eta \\ Y_3 \\ Y_1 \\ Y_2 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{2n-1} \\ \xi_{2n} \end{pmatrix},$$

where  $\Theta =$

$$\begin{pmatrix} 0 & L\omega_3 & L\omega_1 & L\omega_2 & 0 & 0 & 0 & 0 & \cdots & \vec{0} \\ -L\omega_3 & 0 & -L\omega_2 & L\omega_1 & 0 & 0 & 0 & 0 & \cdots & \vec{0} \\ -L\omega_1 & L\omega_2 & 0 & \omega_{12} & \mu\omega_2 & -\mu\omega_1 & 0 & 0 & \cdots & \vec{0} \\ -L\omega_2 & -L\omega_1 & -\omega_{12} & 0 & \mu\omega_1 & \mu\omega_2 & 0 & 0 & \cdots & \vec{0} \\ 0 & 0 & -\mu\omega_2 & -\mu\omega_1 & 0 & \theta_{12} & a_0\omega_2 & a_0\omega_1 & \cdots & \vec{0} \\ 0 & 0 & \mu\omega_1 & -\mu\omega_2 & -\theta_{12} & 0 & -a_0\omega_1 & a_0\omega_2 & \cdots & \vec{0} \\ 0 & 0 & 0 & 0 & -a_0\omega_2 & a_0\omega_1 & 0 & \theta_{34} & \cdots & \vec{0} \\ 0 & 0 & 0 & 0 & -a_0\omega_1 & -a_0\omega_2 & -\theta_{34} & 0 & \cdots & \vec{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vec{0} \\ \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \cdots & B \end{pmatrix}$$

where

$$\mathbf{B} = \begin{pmatrix} -a_{n-2}\omega_2 & a_{n-2}\omega_1 & 0 & \theta_{2n-1\ 2n} \\ -a_{n-2}\omega_1 & -a_{n-2}\omega_2 & -\theta_{2n-1\ 2n} & 0 \end{pmatrix}.$$

Denote the frame as a matrix  $T : M^3 \rightarrow \text{SO}(2n+4)$  with respect to a fixed basis  $\{\mathbf{e}_k\}_{k=1}^{2n+4}$  of  $\mathbb{R}^{2n+4}$ , we can rewrite (8.92) as

$$(8.93) \quad dT = \Theta T.$$

The algebraic form of  $\Theta$  motivates us to introduce a complex structure  $\mathbf{J}$  on  $\mathbb{R}^{2n+4} = \text{Span}_{\mathbb{R}}\{\eta, Y_3, Y_1, Y_2, \xi_1, \xi_2, \dots, \xi_{2n-1}, \xi_{2n}\}$  as below:

$$\mathbf{J} \begin{pmatrix} \eta \\ Y_3 \\ Y_1 \\ Y_2 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{2n-1} \\ \xi_{2n} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ & & & \ddots \\ & & & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \eta \\ Y_3 \\ Y_1 \\ Y_2 \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{2n-1} \\ \xi_{2n} \end{pmatrix}.$$

Denote the diagonal matrix at the right hand side as  $J_0$ . Then the matrix representation of operator  $\mathbf{J}$  under  $\{\mathbf{e}_k\}_{k=1}^{2n+4}$  is:

$$J = T^{-1} J_0 T.$$

Using  $dT = \Theta T$  and the fact that  $J_0$  commutes with  $\Theta$ , it is easy to verify

$$dJ = -T^{-1} dT T^{-1} J_0 T + T^{-1} J_0 dT = -T^{-1} \Theta J_0 T + T^{-1} J_0 \Theta T = 0.$$

So  $\mathbf{J}$  is a well-defined complex structure on this  $\mathbb{R}^{2n+4}$ .

Another way to look at the structure equations (8.92) is to consider the complex version. We define

$$\mathcal{Z}_1 = \eta + iY_3, \mathcal{Z}_2 = Y_1 + Y_2, \zeta_1 = \xi_1 - i\xi_2, \dots, \zeta_n = \xi_{2n-1} - i\xi_{2n}.$$

Then the complex version of the equation (8.92) is

$$(8.94) \quad \begin{aligned} d\mathcal{Z}_1 &= -iL\omega_3\mathcal{Z}_1 + L(\omega_1 - i\omega_2)\mathcal{Z}_2, \\ d\mathcal{Z}_2 &= -L(\omega_1 + i\omega_2)\mathcal{Z}_1 - i\omega_{12}\mathcal{Z}_2 + i\mu(\omega_1 - i\omega_2)\zeta_1, \\ d\zeta_1 &= i\mu(\omega_1 + i\omega_2)\mathcal{Z}_2 + i\theta_{12}\zeta_1 + ia_0(\omega_1 - i\omega_2)\zeta_2, \\ d\zeta_k &= ia_{k-2}(\omega_1 + i\omega_2)\zeta_{k-1} + i\theta_{2k-1\ 2k}\zeta_k \\ &\quad + ia_{k-1}(\omega_1 - i\omega_2)\zeta_{k+1}, \quad 2 \leq k \leq n-1, \\ d\zeta_n &= ia_{n-2}(\omega_1 + i\omega_2)\zeta_{n-1} + i\theta_{2n-1\ 2n}\zeta_n. \end{aligned}$$

Geometrically, this implies that

$$\mathbb{C}^{n+2} = \text{Span}_{\mathbb{C}}\{\mathcal{Z}_1, \mathcal{Z}_2, \zeta_1, \zeta_2, \dots, \zeta_n\},$$

is a fixed  $n+2$  dimensional complex vector space endowed with the complex structure  $i$ , which is identified with  $(\mathbb{R}^{2n+4}, \mathbf{J})$  via the following isomorphism between complex linear spaces:

$$v \in \mathbb{C}^{n+2} \mapsto \text{Re}(v) \in \mathbb{R}^{2n+4}.$$

For example,  $\eta + iY_3 \mapsto \eta, i\eta - Y_3 \mapsto -Y_3$  and so on.

The second geometrical conclusion is an interpretation of (??) that  $[\eta + iY_3]$  defines a holomorphic mapping from the quotient surface  $\overline{M}^2 = M^3/\Gamma$  to the projective space  $\mathbb{C}P^{n+1}$ . Moreover, the unit circle in

$$\text{Span}_{\mathbb{R}}\{\eta, Y_3\} = \text{Span}_{\mathbb{C}}\{\eta + iY_3\}$$

is a fiber of the Hopf fibration of  $\mathbb{S}^{2n+3} \subset (\mathbb{R}^{2n+4}, \mathbf{J})$ . In fact it corresponds to the subspace  $\text{Span}_{\mathbb{R}}\{Y, \hat{Y}, Y_3\}$ , which is geometrically a leave of the foliation  $(M^3, \Gamma)$ . To see this, it follows from the equations of  $d(\xi_{2k-1} - i\xi_{2k})$  that  $\{\xi_1, \xi_2, \dots, \xi_{2n}, d\xi_1, d\xi_2, \dots, d\xi_{2n}\}$  span a  $(2n+2)$ -dimensional spacelike subspace, the corresponding 2-parameter family of 3-dimensional mean curvature sphere congruence has an envelop  $M^3$ , whose points correspond to the light-like directions in the orthogonal complement  $\text{Span}\{Y, \hat{Y}, Y_3\}$ . In particular  $[Y], [\hat{Y}]$  are two points on this circle and such circles form a 2-parameter family, with  $\overline{M}$  as the parameter space, they give a foliation of  $M^3$  which is also a circle fibration. Thus the whole  $M^3$  is the Hopf lift of  $\overline{M}^2 \rightarrow \mathbb{C}P^{n+1}$ . In other words we have the following commutative diagram

$$\begin{array}{ccccc} M^3 & \xrightarrow{\eta} & \mathbb{S}^{2n+3} & \xrightarrow{\subset} & \mathbb{C}^{n+2} \\ & \searrow [\eta+iY_3] & \downarrow \pi & \swarrow \pi & \\ M^3/\Gamma & & \overline{M}^2 & \xrightarrow{\quad} & \mathbb{C}P^{n+1} \end{array}.$$

Next we prove  $\overline{M}^2 \rightarrow \mathbb{C}P^{n+1}$  is the Veronese surface of  $\mathbb{C}P^{n+1}$ . Note that any fibre of the Hopf fibration has the  $\mathbb{S}^1$  homogenous. So  $\overline{M}^2 \rightarrow \mathbb{C}P^{n+1}$  must be homogenous and hence has constant curvature. The conclusion follows from the classical results of Calabi([7] [1]).  $\square$

Combining Proposition 8.1, Proposition 7.3 and Proposition 8.2, we finish the proof of our main Theorem 1.6.

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